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ANALYSIS OF THE NUMERICAL SOLUTION OF THE SHALLOW WATER EQUATIONS

by

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This thesis is concerned with the analysis of various methods for the numerical solution of the shallow water equations along with the stability of these methods. Most of the thesis is concerned with the background and formulation of the shallow water equations. The derivation of the basic equations will be given, in the primitive variable and vorticity-divergence formulation. Also the shallow water equations will be written in spherical coordinates. Two main types of methods used in approximating differential equations of this nature will be discussed. The two schemes are finite difference method (FDM) and the finite element method (FEM). After presenting the shallow water equations in several formulations, some examples will be presented. The use of the Fourier transform to find the solution of a semidiscrete analog of the shallow water equations is also demonstrated.

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ANALYSIS OF THE NUMERICAL SOLUTION OF THE SHALLOW WATER EQUATIONS

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This thesis is concerned with the analysis of various methods for the numerical solution of the shallow water equations along with the stability of these methods. Most of the thesis is concerned with the background and formulation of the shallow water equations. The derivation of the basic equations will be given, in the primitive variable and vorticity-divergence formulation. Also the shallow water equations will be written in spherical coordinates. Two main types of methods used in approximating differential equations of this nature will be discussed. The two schemes are finite difference method (FDM) and the finite element method (FEM). After presenting the shallow water equations in several formulations, some examples will be presented. The use of the Fourier transform to find the solution of a semidiscrete analog of the shallow water equations is also demonstrated.

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I would like to thank my thesis advisor and the faculty and staff of Naval Postgraduate School for giving me the time and instruction to work in this field of mathematics. I am especially thankful that I had a chance to work with Professor Gordon Latta. I am also in debt to the USN for allowing me to come to this prestigious academic environment. My gratitude also goes out to NCSU for providing me with the basic tools which helped make my time here more successful. I am forever in debt to my parents, family and friends for providing me the courage and support to continue learning. Their encouragement of my pursuit of excellence has helped me to achieve a wide range of goals in my life. I sincerely hope that my knowledge and character will lend support and encouragement to all with whom I am involved in the future. Again, thank you.

Thomas A. Hamrick

I. INTRODUCTION

The majority of this work is a compilation of past mathematical papers concerning the numerical solution of the shallow water equations, and I can in no way take credit for them. Hopefully I have put this material in a readable and understandable text that will provide a good basis for others to use for reference in this area of mathematics. The next several paragraphs will highlight the key points in slightly more detail than the Table of Contents.

Chapter II shall review and discuss the mathematical formulation of flow in shallow regions. Two formulations will be given, namely the primitive and the vorticity-divergence formulations of the shallow water equations. It will also look at the formulation of the equations in spherical coordinates.

Chapter III will look at the linearization of the shallow water equations. In Cartesian coordinates we discuss the one dimensional and two dimensional cases. The linearized vorticity-divergence form of the shallow water equations is derived in this chapter for the two dimensional case. The spherical case is also discussed here. Note that in the Cartesian case, the linearized equations have constant coefficients in contrast to the spherical coordinates. This means that linear stability analysis will be more difficult in the latter.

Chapter IV shall develop two of the different types of approximations commonly used. The first is the finite difference, and the second is the finite element. The finite element may also be referenced to as the Galerkin method and will be referred to by both names in this thesis. Other families of methods such as spectral or finite volume approximation are beyond the scope of this thesis.

In Chapter V, the stability analysis of the shallow water equations will be examined. For this analysis we use the linearized equations obtained in Chapter II. Fourier transforms in a one dimensional case will be covered. Fourier transforms will also be used to look at the two dimensional case. The spherical case requires

special consideration as in Longuet-Higgins(see [Ref. 1]) or Neta (see [Ref. 2]). A summary of the results for the various techniques are provided in a table for ease of later examination.

Also covered in this report will be an extensive bibliography and reference section, so that future students studying in this area will be able to pick up more easily where this thesis leaves off.

II. SHALLOW WATER MODEL

A. MODEL BACKGROUND

For this model consider a sheet of fluid with constant and uniform density. (See [Ref. 3].) The height of the surface of the fluid above the reference level $z = 0$ is $h(x, y, t)$. We model the body force arising from the potential $\vec{\phi} = \vec{g}h$ with atmosphere or ocean in mind. $\vec{\phi}$ is a vector directed perpendicular to the $z = 0$ surface, or $\vec{\phi}$ can be said to be antiparallel to the vertical axis (i.e., $\vec{\phi}$ is in the direction opposite to the vertical axis). The rotation axis of the fluid is the z-axis in this model. In this case the Coriolis parameter f is $2\vec{\Omega}$ since $\vec{\Omega} = \vec{k}\Omega$. The rigid bottom is defined by the surface $z = h_B(x, y)$. The velocity has components u, v, and w in the x, y, and z directions respectively. Though the pressure of the fluid surface can be arbitrarily imposed, for this model it will be assumed to be constant. Lastly, the fluid is assumed *inviscid*, in other words, only the motions for which viscosity is not important are considered.

In this model, because the depth of the fluid, $h - h_B$, varies over time or space, let H be the average depth of the fluid. H characterizes the vertical scale of motion also. Let L be the characteristic horizontal scale for the motion. Then a fundamental condition which will characterize shallow water theory will be

$$\frac{H}{L} \ll 1$$

which is also called the hydrostatic approximation with long wavelengths. (See [Ref. 4].) The shallow water model contains several of the important dynamical features of the atmosphere and ocean while being simple enough to be easily understood. The major physical difference with this model and reality is the absence of density stratification that is present in the real fluids such as Earth's atmosphere or oceans. The hydrostatic approximation also allows ρ to vary with z, but we will consider ρ constant for this model.

Recalling the equation for motions of fluids from Haltiner and Williams (see [Ref. 5], we have

$$\frac{d\vec{V}}{dt} = -\alpha\vec{\nabla}p - 2\vec{\Omega} \times \vec{V} + \vec{g} + \vec{F}. \tag{2.1}$$

α is the specific gravity (i.e. $\frac{1}{\rho}$), $\vec{g}$ is the sum of gravitational and centrifugal forces, and $\vec{F}$ is the force due to friction. For our model we will be neglecting $\vec{g}$ and $\vec{F}$ because with our assumptions they are much smaller in magnitude than the Coriolis forces. Recall that

$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + \frac{1}{2}\vec{\nabla}(\vec{V} \cdot \vec{V}) + (\vec{\nabla} \times \vec{V}) \times \vec{V}, \tag{2.2}$$

where $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. Using these two basic equations, we will construct our model.

B. MODEL EQUATIONS

There are several formulations in the literature. We will cover

- primitive variable formulation
- vorticity divergence formulation.

The formulations using stream function and velocity potential will not be discussed here, (see, for example, [Ref. 5]).

1. Primitive Form

Now follow the consequences of the model in the realm of the dynamical equations of motion. Recalling the Navier-Stokes equations which describe the conservation of mass and momentum (see [Ref. 6]), we see that the dynamics and the thermodynamics decouple due to the specification of incompressibility and constant

density. (See [Ref. 3].) This reduces the equation of *mass conservation* to the condition of incompressibility:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.3}$$

Now the first two terms of this equation are of order $\frac{U}{L}$, where U can be considered the characteristic scale for the horizontal velocity. It then follows that the scale for the vertical velocity (W) is smaller than or equal to the order δU. This represents an upper bound for the vertical velocity, and it can be *smaller* than order δU if there is cancellation between $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Now an estimate of the momentum equations in component form are:

$$\begin{array}{cccccccc}
\frac{\partial u}{\partial t} & + u\frac{\partial u}{\partial x} & + v\frac{\partial u}{\partial y} & + w\frac{\partial u}{\partial z} & - fv & = & -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} \\
\frac{U}{T} & \frac{U^2}{L} & \frac{U^2}{L} & \frac{UW}{H} & fU & & \frac{P}{\rho L} \\
\\
\frac{\partial v}{\partial t} & + u\frac{\partial v}{\partial x} & + v\frac{\partial v}{\partial y} & + w\frac{\partial v}{\partial z} & + fu & = & -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial y} \\
\frac{U}{T} & \frac{U^2}{L} & \frac{U^2}{L} & \frac{UW}{H} & fU & & \frac{P}{\rho L} \\
\\
\frac{\partial w}{\partial t} & + u\frac{\partial w}{\partial x} & + v\frac{\partial w}{\partial y} & + w\frac{\partial w}{\partial z} & & = & -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} \\
\frac{W}{T} & \frac{UW}{L} & \frac{UW}{L} & \frac{W^2}{H} & & & \frac{P}{\rho H}
\end{array} \tag{2.4}$$

f will be defined differently depending on the form of the shallow water equations. In (2.4) each term has the order of magnitude written immediately below it in terms of the characteristic scales where T is the characteristic scale for time and P is the characteristic scale for the pressure field. Equation (2.4) follows from (2.1) and (2.2) in the following manner. Remember that we are neglecting the gravity term and the

force due to friction. Now when we expand out (2.1), we get

$$\begin{aligned}\frac{d\vec{V}}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\vec{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\vec{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\vec{k} \\ &\quad +2\Omega(\vec{j}\cos\varphi + \vec{k}\sin\varphi) \times (u\vec{i} + v\vec{j} + w\vec{k}),\end{aligned}$$

which equals

$$\begin{aligned}\frac{d\vec{V}}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\vec{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\vec{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\vec{k} \\ &\quad +(2w\Omega\cos\varphi - fv)\vec{i} + fu\vec{j} - 2u\Omega\cos\varphi\vec{k}.\end{aligned} \tag{2.5}$$

Since we are modeling a rectangular system first, the φ terms drop out and (2.5) becomes

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\vec{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\vec{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\vec{k} - fv\vec{i} + fu\vec{j}. \tag{2.6}$$

When we look at the spherical form of the shallow water equations, the φ terms are used. Note that

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}.$$

Now, expanding out (2.2) we get

$$\begin{aligned}\frac{d\vec{V}}{dt} &= \frac{\partial \vec{V}}{\partial t} + \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right)\vec{i} + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y}\right)\vec{j} \\ &\quad + \left(u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z}\right)\vec{k} + \left[w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]\vec{i} \\ &\quad + \left[u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - w\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\right]\vec{j} + \left[v\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - u\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\right]\vec{k}.\end{aligned}$$

Collecting terms, this becomes

$$\begin{aligned}\frac{d\vec{V}}{dt} &= \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\vec{i} \\ &\quad + \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\vec{j} + \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\vec{k}\end{aligned}$$

or more simply

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot D).$$

Equating the right hand side of (2.6) and this equation and then separating into the three components will give us (2.4).

Note that the total pressure (p) is

$$p(x, y, z, t) = -\rho g z + \tilde{p}(x, y, z, t).$$

The horizontal pressure gradient is independent of z if one uses the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -\rho g + \mathbf{O}(\delta^2) \qquad (2.7)$$

where δ is $\frac{H}{L}$ and $\delta \ll 1$. This approximation follows from the scale analysis of the momentum equations and the incompressibility condition. Integrating (2.7) and using the boundary condition

$$p(x, y, h) = p_0$$

yields

$$p = \rho g(h - z) + p_0.$$

A way to look at this is to think of the pressure which is in excess of p_0 at any point simply as the weight of the unit column of fluid above that point at that particular instant in time. Remember that the *horizontal pressure gradient* is independent of z, and therefore the horizontal accelerations are independent of z.

The Rossby-wave number is the ratio of inertia to the Coriolis terms or $(\frac{U}{fL})$. Recalling the Taylor-Proudman theorem (see, [Ref. 3] p. 43) which simply put states that if the Rossby-wave number is small, friction can be ignored, and baroclinic vector is identically zero (i.e., there is no additional pressure change with a change in height as in this model), then it follows that the horizontal accelerations are independent of z

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0.$$

In this model, the Taylor-Proudman theorem applied to a homogeneous fluid *requires* the velocities to be independent of z. This allows the horizontal momentum equations to become

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial y}
\end{aligned} \tag{2.8}$$

Let us look at the scale analysis of the vertical momentum equation of (2.4). Consider $\tilde{w}$ to be of $\mathbf{O}\left(\frac{H}{L}\right) = \mathbf{O}(\delta)$. Scale u and v by U, and w by $\frac{H}{L}U$. By (2.3) we know that

$$\frac{\partial \tilde{u}}{\partial x'} + \frac{\partial \tilde{v}}{\partial y'} + \frac{\partial \tilde{w}}{\partial z'} = 0.$$

where

$$\begin{aligned}
u &= U\tilde{u} & x' &= \frac{x}{L} \\
v &= U\tilde{v} & y' &= \frac{y}{L} \\
w &= \delta U\tilde{w} & z' &= \frac{z}{H}.
\end{aligned}$$

The vertical momentum equation can be written as

$$\delta U\left[\frac{\partial \tilde{w}}{\partial t} + \frac{U}{L}\tilde{u}\frac{\partial \tilde{w}}{\partial x'} + \frac{U}{L}\tilde{v}\frac{\partial \tilde{w}}{\partial y'} + \delta\frac{U}{H}\tilde{w}\frac{\partial \tilde{w}}{\partial z'}\right] = -\frac{1}{H}\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z'}.$$

Multiply through by H and this becomes

$$\delta U\left[H\frac{\partial \tilde{w}}{\partial t} + U\delta\tilde{u}\frac{\partial \tilde{w}}{\partial x'} + U\delta\tilde{v}\frac{\partial \tilde{w}}{\partial y'} + \delta U\tilde{w}\frac{\partial \tilde{w}}{\partial z'}\right] = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z'}.$$

Removing all the terms of $\mathbf{O}(\delta^2)$, note from (2.7) that $\tilde{p}$ is of $\mathbf{O}(\delta^2)$ also, we have

$$\delta U H\frac{\partial \tilde{w}}{\partial t} = 0.$$

Therefore the vertical momentum equation is an identity.

Now we will rewrite the incompressibilty condition (2.3),

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Since the horizontal accelerations are independent of z, we can integrate the above relationship to get

$$w(x,y,z,t) = -z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \tilde{w}(x,y,t). \tag{2.9}$$

The condition of *no* normal flow at the bottom requires

$$w(x,y,h_B,t) = u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y}.$$

It naturally follows that

$$w(x,y,z,t) = (h_B - z)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y}.$$

The vertical velocity $w(x,y,z,t) = \dfrac{dz}{dt}$ at the upper surface $z = h$ represents the rate at which the free surface is rising. Thus $w(x,y,h,t) = \dfrac{dh}{dt}$ and (2.9) become

$$w(x,y,h,t) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}.$$

Therefore

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[u(h - h_B)\right] + \frac{\partial}{\partial y}\left[v(h - h_B)\right] = 0.$$

This equation, combined with the horizontal momentum equations (2.9) form the shallow water equations.

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[u(h - h_B)\right] + \frac{\partial}{\partial y}\left[v(h - h_B)\right] &= 0.
\end{aligned} \tag{2.10}$$

u, v, and h are called the *primitive variables* and (2.10) is the *primitive form* of the shallow water equations.

2. Vorticity-divergence Form

For vorticity-divergence, we start with the definitions. The relative vorticity, denoted by ζ, is defined by

$$\zeta = v_x - u_y, \tag{2.11}$$

and the absolute vorticity, denoted by Q, is given by

$$Q = \zeta + f. \tag{2.12}$$

The divergence, denoted by D, is given by

$$D = u_x + v_y, \tag{2.13}$$

and the kinetic energy (per unit mass), denoted by K, is given by

$$K = \frac{u^2 + v^2}{2}. \tag{2.14}$$

To get the vorticity-divergence formulation, we start by differentiating the first equation of (2.10) with respect to y and subtract it from the second equation of (2.10) differentiated with respect to x gives

$$\begin{aligned} &v_{xt} + u_x v_x + u v_{xx} + v_x v_y + v v_{xy} + f u_x + g h_{xy} \\ &-u_{yt} - u_y u_x - u u_{xy} - v_y u_y - v u_{yy} + f v_y - g h_{xy} = 0. \end{aligned} \tag{2.15}$$

Rearranging and cancelling terms will yield

$$\begin{aligned} &(v_x - u_y)_t + u_x(v_x - u_y) + u(v_x - u_y)_x \\ &+v_y(v_x - u_y) + v(v_x - u_y)_y + f(u_x + v_y) = 0. \end{aligned}$$

Rewriting in terms of (2.11) and (2.13) gives

$$\zeta_t + u_x\zeta + u\zeta_x + v_y\zeta + v\zeta_y + fD = 0.$$

This can be rewritten in the form

$$\zeta_t + (u\zeta)_x + (v\zeta)_y + fD = 0$$

or

$$\frac{d\zeta}{dt} + fD = 0$$

This is called the vorticity equation. To get the divergence equation, we diffferentiate the first equation of (2.10) with respect to x and add to it the second equation of (2.10) differentiated with respect to y gives

$$\begin{aligned} &u_{xt} + u_x{}^2 + uu_{xx} + v_x u_y + vu_{xy} - fv_x + gh_{xx} \\ &+v_{yt} + u_y v_x + uv_{xy} + v_y{}^2 + vv_{yy} + fu_y + gh_{yy} = 0. \end{aligned} \tag{2.16}$$

Again, rearranging terms will yield

$$\begin{aligned} &(u_x + v_y)_t + u(u_x + v_y)_x + v(u_x + v_y)_y + \\ &u_x{}^2 + v_y{}^2 + 2u_y v_x - f(v_x - u_y) + g(h_{xx} + h_{yy}) = 0. \end{aligned}$$

Using (2.11) and (2.13) this can be further reduced to

$$D_t + uD_x + vD_y + u_x{}^2 + v_y{}^2 + 2u_y v_x - f\zeta + g\nabla^2 h = 0.$$

The vorticity-divergence form of the shallow water equations in terms of relative vorticity and divergence are

$$\begin{aligned} \zeta_t + (u\zeta)_x + (v\zeta)_y + fD &= 0 \\ D_t + uD_x + vD_y + u_x{}^2 + v_y{}^2 + 2u_y v_x - f\zeta + g\nabla^2 h &= 0 \\ h_t + [u(h - h_B)]_x + [v(h - h_B)]_y &= 0. \end{aligned} \tag{2.17}$$

To simplify a bit further, we can use (2.11) through (2.14) on (2.15) and (2.13). First, rearrange and cancel terms of (2.15) in the following manner.

$$\begin{aligned} &v_{xt} - u_{yt} + u_x v_x - u_y u_x + fu_x + uv_{xx} - uu_{yx} + \\ &v_y v_x - v_y u_y + fv_y + vv_{xy} - vu_{yy} = 0 \end{aligned}$$

Rearranging further gives

$$\begin{aligned} &(v_x - u_y)_t + u_x(v_x - u_y + f) + u(v_{xx} - u_{yx}) + \\ &v_y(v_x - u_y + f) + v(v_{xy} - u_{yy}) = 0. \end{aligned}$$

This leads to

$$\zeta_t + u_x Q + u Q_x + v_y Q + v Q_y = 0,$$

or

$$\zeta_t + (uQ)_x + (vQ)_y = 0. \tag{2.18}$$

This can also be written as

$$Q_t + (uQ)_x + (vQ)_y = 0$$

since $f_t = 0$.

Now we simplify (2.16) in a similar fashion. First, rewrite the equation in the following manner.

$$\begin{aligned}
&u_{xt} + v_{yt} + gh_{xx} + gh_{yy} + uu_{xx} + u_x u_x + vv_{xx} + v_x v_x + \\
&uu_{yy} + u_y u_y + vv_{yy} + v_y v_y - v_x v_x + v_x u_y - f v_x - vv_{xx} + \\
&vu_{yx} + u_y v_x - u_y u_y + f u_y + uv_{xy} - uu_{yy} = 0
\end{aligned}$$

Notice that we added in and subtracted some terms that were alike. Now reduce this equation to

$$\begin{aligned}
&u_{xt} + v_{yt} + gh_{xx} + gh_{yy} + (uu_x + vv_x)_x + \\
&(uu_y + vv_y)_y - v_x v_x + v_x u_y - f v_x - vv_{xx} + \\
&vu_{yx} + u_y v_x - u_y u_y + f u_y + uv_{xy} - uu_{yy} = 0.
\end{aligned}$$

Reorganize the terms to further reduce to

$$\begin{aligned}
&u_{xt} + v_{yt} + gh_{xx} + gh_{yy} + \left(\frac{u^2 + v^2}{2}\right)_{xx} + \\
&\left(\frac{u^2 + v^2}{2}\right)_{yy} - v_x(v_x - u_y + f) - v(v_x - u_y + f)_x + \\
&u_y(v_x - u_y + f) + u(v_x - u_y + f)_y = 0.
\end{aligned}$$

Now use the definitions of Q and K to reduce this equation to

$$\begin{aligned}
&(u_x + v_y)_t + g(h_{xx} + h_{yy}) + K_{xx} + K_{yy} - \\
&v_x Q - vQ_x + u_y Q + uQ_y = 0.
\end{aligned}$$

Thus our equation becomes

$$D_t + g\nabla^2 h + \nabla^2 K - (vQ)_x + (uQ)_y = 0. \tag{2.19}$$

Combining (2.18), (2.19), and the last equation of (2.10) we have the shallow water equations in the vorticity divergence form.

$$\begin{aligned} \zeta_t + (uQ)_x + (vQ)_y &= 0 \\ D_t + g\nabla^2 h + \nabla^2 K - (vQ)_x + (uQ)_y &= 0 \\ h_t + [u(h - h_B)]_x + [v(h - h_B)]_y &= 0 \end{aligned} \tag{2.20}$$

C. SPHERICAL COORDINATES

1. Background

Where Cartesian coordinates locate a point using 3 vectors, spherical coordinates locate the same point using two angles and a distance. Typically, spherical coordinates represent a point in space with an ordered triplet such as (a, ϕ, λ). These variables are related to (x, y, z) by

$$\begin{aligned} x &= a \sin\phi \cos\lambda \\ y &= a \sin\phi \sin\lambda \\ z &= a \cos\phi. \end{aligned} \tag{2.21}$$

Let us consider a channel that goes around the whole Earth. In this model, let λ be the longitude, let ϕ be the latitude, let r be the radial distance, let a be the average radius of the Earth, and let z be the average sea level. Now, recall equations (2.5) and (2.2).

$$\begin{aligned} \frac{d\vec{V}}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\vec{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\vec{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\vec{k} \\ &\quad +(2w\Omega\cos\phi - fv)\vec{i} + fu\vec{j} - 2u\Omega\cos\phi\vec{k}. \end{aligned}$$

and

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\vec{\nabla}(\vec{V}\cdot\vec{V}) + (\vec{\nabla}\times\vec{V})\times\vec{V}.$$

Using results drawn from Haltiner and Williams (see [Ref. 5]), and neglecting the force due to friction, F, gives us

$$\begin{aligned}
0 \;=\;& \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\vec{i} \\
&+ \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\vec{j} + \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\vec{k} \\
&+ \left\{\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(2\Omega + \frac{u}{a\cos\phi}\right)(v\sin\phi - w\cos\phi)\right\}\vec{i} \qquad (2.22) \\
&+ \left\{\frac{1}{\rho}\frac{\partial p}{\partial y} + \left(2\Omega + \frac{u}{a\cos\phi}\right)u\sin\phi - \frac{vw}{a}\right\}\vec{j} \\
&+ \left\{\frac{1}{\rho}\frac{\partial p}{\partial z} + g + \left(2\Omega + \frac{u}{a\cos\phi}\right)u\cos\phi + \frac{v^2}{a}\right\}\vec{k}
\end{aligned}$$

Separating along the three axes and we have

$$\begin{aligned}
0 &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} - \left(2\Omega + \frac{u}{a\cos\phi}\right)(v\sin\phi - w\cos\phi) \\
0 &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \left(2\Omega + \frac{u}{a\cos\phi}\right)u\sin\phi - \frac{vw}{a} \qquad (2.23) \\
0 &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g + \left(2\Omega + \frac{u}{a\cos\phi}\right)u\cos\phi + \frac{v^2}{a}
\end{aligned}$$

Note that p is independent of z, and $\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $\frac{1}{\rho}\frac{\partial p}{\partial y}$ can be written in terms of $g\frac{\partial h}{\partial x}$ and $g\frac{\partial h}{\partial y}$ respectively. Also note that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \lambda}\frac{\partial \lambda}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} \\
&= \frac{\partial u}{\partial \lambda}\frac{1}{-a\sin\phi\sin\lambda} + \frac{\partial u}{\partial \phi}\frac{1}{a\cos\phi\cos\lambda}
\end{aligned}$$

and similarly for the other partials. Substituting these and the Coriolis parameter, $f = 2\vec{\Omega}\sin\theta$, into (2.23) gives us the shallow water equations in spherical coordinates. (See [Ref. 7].)

$$\frac{\partial u}{\partial t} + \frac{1}{a\cos\theta}\left[u\frac{\partial u}{\partial \lambda} + v\cos\theta\frac{\partial u}{\partial \theta}\right] - \left[f + \frac{u}{a}\tan\theta\right]v + \frac{g}{a\cos\theta}\frac{\partial h}{\partial \lambda} = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{a\cos\theta}\left[u\frac{\partial v}{\partial \lambda} + v\cos\theta\frac{\partial v}{\partial \theta}\right] + \left[f + \frac{u}{a}\tan\theta\right]u + \frac{g}{a}\frac{\partial h}{\partial \theta} = 0 \tag{2.24}$$

$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\theta}\left[\frac{\partial}{\partial \lambda}(hu) + \frac{\partial}{\partial \theta}(hv\cos\theta)\right] = 0.$$

Numerical approximations of the shallow water equations in spherical coordinates are given for example by Turkel and Zwas (see [Ref. 8]) and Arakawa (see [Ref. 9]) et al. Analysis of the schemes were given by Neta and Navon (see [Ref. 10]) et al. Navon and de Villiers et al. have applied the Turkel-Zwas scheme to a hemispheric barotropic model (see [Ref. 11]).

III. LINEARIZATION

In this chapter, we will see how to linearize the shallow water equations in Cartesian and spherical coordinates. The Cartesian coordinate case starts with either the primitive or the vorticity-divergence formulation.

A. TWO DIMENSIONS

1. Basic Formula

First separate each of the three variables into the average which we can consider constant plus the perturbation in the variable. This should look like:

$$\begin{aligned} u &= U + u' \\ v &= V + v' \\ h &= H + h' \end{aligned} \tag{3.1}$$

where the capital letters are used for the average or mean values and the primed variables are the perturbations. Now substituting this into (2.7) and omitting all the second order terms gives

$$\begin{aligned}
\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + V\frac{\partial u'}{\partial y} - fV - fv' + g\frac{\partial (H+h')}{\partial x} &= 0 \\
\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y} + fU + fu' + g\frac{\partial (H+h')}{\partial y} &= 0 \\
\frac{\partial h'}{\partial t} + \frac{\partial}{\partial x}\left[u'(H-h_B)\right] + \frac{\partial}{\partial y}\left[v'(h-h_B)\right] + (U+V)\frac{\partial}{\partial x}\left[H+h'-h_B\right] &= 0.
\end{aligned} \tag{3.2}$$

For simplicity and without loss of generality we sometimes consider a zero mean flow, i.e. $U = V = 0$. This and dropping the primes will give the linearized form of the shallow water equations which follows.

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial h}{\partial t} + (H - h_B)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

If topography is not considered, then the linearized shallow water equations can be written

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

This can also be written as

$$\begin{aligned} u_t - fv + gh_x &= 0 \\ v_t + fu + gh_y &= 0 \\ h_t + H(u_x + v_y) &= 0. \end{aligned} \tag{3.3}$$

Another source for the development of the linearized shallow water equations is Gill. (See [Ref. 4]).

2. Vorticity Divergence Formula

The two dimensional linearized vorticity divergence form of the shallow water equations is obtained by using (2.8) and (2.9) on (3.3). It should be noted that the linearized vorticity divergence formula can be found linearizing (2.16) or (2.19). Taking the partial differential of the first equation of (3.3) with respect to y and subtracting it from the partial differential of the second equation of (3.3) with respect to x yields

$$v_{tx} + fu_x + gh_{xy} - u_{ty} + fv_y - gh_{xy} = 0.$$

With a little algebra this equation can be rewritten as

$$(v_x - u_y)_t + f(u_x + v_x) = 0.$$

which when using the definitions of ζ and D reduces to

$$\zeta_t + fD = 0. \tag{3.4}$$

This is the first equation of the linearized vorticity divergence form. By taking the partial of the first equation of (3.3) with respect to x and adding it to the partial differential of the second equation of (3.3) with respect to y we get

$$u_{tx} - fv_x + gh_{xx} + v_{ty} + fu_y + gh_{yy} = 0.$$

With a little algebraic simplification our equation becomes

$$(u_x + v_y)_t - f(v_x - u_y) + g(h_{xx} + h_{yy}) = 0.$$

Again, using the definitions of ζ and D, this reduces to

$$D_t - f\zeta + g\nabla^2 h = 0. \tag{3.5}$$

(3.4) and (3.5) along with the last equation of (3.3) form the two dimensional linearized vorticity divergence form of the shallow water equations.

$$\begin{aligned} \zeta_t + fD &= 0 \\ D_t - f\zeta + g\nabla^2 h &= 0 \\ h_t + H(u_x + v_y) &= 0 \end{aligned} \tag{3.6}$$

B. LINEARIZED SHALLOW WATER EQUATIONS IN ONE DIMENSION

In the one dimensional case, all dependence on y is eliminated and we end up with

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu = 0$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x}\right) = 0,$$

or more simply

$$\begin{aligned} u_t - fv + gh_x &= 0 \\ v_t + fu &= 0 \\ h_t + Hu_x &= 0. \end{aligned} \tag{3.7}$$

C. SPHERICAL

To linearize the spherical version of the shallow water equations, simply take (3.1) and substitute it into (2.26). The resulting equation follows quite easily once all second and higher order terms are omitted.

$$\frac{\partial u}{\partial t} - fv + \frac{g}{a\cos\theta}\frac{\partial h}{\partial \lambda} = 0$$

$$\frac{\partial v}{\partial t} + fu + \frac{g}{a}\frac{\partial h}{\partial \theta} = 0$$

$$\frac{\partial h}{\partial t} + \frac{H}{a\cos\theta}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \theta}(v\cos\theta)\right) = 0.$$

Notice that the linearized equations have nonconstant coefficients. This complicates the linear stabilty analysis. See Longuet-Higgins [Ref. 1] and Neta [Ref. 12] et al.

IV. APPROXIMATIONS

A. FINITE DIFFERENCES

1. Introduction

One of the first steps in using finite difference methods is to replace the continuous problem domain by a difference mesh or a grid. Let $f(x)$ be a function of the single independent variable x for $a \le x \le b$. The interval $[a,b]$ is discretized by considering the nodes $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$, and we denote $f(x_i)$ by f_i. The mesh size is $x_{i+1} - x_i$ and we shall assume for simplicity that the mesh size is a constant

$$h = \frac{b-a}{N+1}$$

and

$$x_i = a + ih \qquad i = 0, 1, \cdots, N+1$$

In the two dimensional case, the function $f(x,y)$ may be specified at nodal point (x_i, y_j) by f_{ij}. The spacing in the x direction is h_x and in the y direction is h_y.

Taylor series expansions of functions in several variables play an important role in formulation and classification of finite difference methods. The Taylor series expansion for f_{i+1} about the point x_i is given by

$$f_{i+1} = f_i + hf_i' + \frac{h^2}{2!}f_i'' + \frac{h^3}{3!}f_i'' + \cdots$$

The Taylor series expansion for $f_{i+1\,j+1}$ about the point (x_i, y_j) is given by

$$f_{i+1\,j+1} = f_{ij} + (h_x f_x + h_y f_y)_{ij} + (\frac{h_x^2}{2}f_{xx} + h_x h_y f_{xy} + \frac{h_y^2}{2}f_{yy})_{ij} + \cdots$$

2. Finite Differences

Using text drawn from Neta (see [Ref. 13]), we will cover certain aspects of finite differencing. An infinite number of difference representations can be found for the partial derivatives of $f(x,y)$. Let us use the following operators:

$$\Delta_x f_{ij} = f_{i+1\,j} - f_{ij} \quad 1^{st} \text{ forward difference operator}$$

$$\nabla_x f_{ij} = f_{ij} - f_{i-1j} \quad 1^{st} \text{ backward difference operator}$$

$$\bar{\delta}_x f_{ij} = f_{i+1j} - f_{i-1j} \quad \text{centered difference}$$

$$\delta_x f_{ij} = f_{i+1/2j} - f_{i-1/2j}$$

$$\mu_x f_{ij} = (f_{i+1/2j} + f_{i-1/2j})/2 \quad \text{averaging operator}$$

Note that

$$\bar{\delta}_x = 2\mu_x \delta_x.$$

In a similar fashion we can define the corresponding operators in y.

In the following table we collected some of the common approximations for the first derivative.

Table I. First Derivative Approximations

Finite Difference	Order (See Chapter IV Section A 3)
$\frac{1}{h_x}\Delta_x f_{ij}$	$O(h_x)$
$\frac{1}{h_x}\nabla_x f_{ij}$	$O(h_x)$
$\frac{1}{2h_x}\bar{\delta}_x f_{ij}$	$O(h_x^2)$
$\frac{1}{2h_x}(-3f_{ij} + 4f_{i+1j} - f_{i+2j}) = \frac{1}{h_x}(\Delta_x - \frac{1}{2}\Delta_x^2) f_{ij}$	$O(h_x^2)$
$\frac{1}{2h_x}(3f_{ij} - 4f_{i-1j} + f_{i-2j}) = \frac{1}{h_x}(\nabla_x + \frac{1}{2}\nabla_x^2) f_{ij}$	$O(h_x^2)$
$\frac{1}{h_x}(\mu_x\delta_x - \frac{1}{3!}\mu_x\delta_x^3) f_{ij}$	$O(h_x^3)$
$\frac{1}{2h_x}\frac{\bar{\delta}_x f_{ij}}{1 + \frac{1}{6}\delta_x^2}$	$O(h_x^4)$

The compact fourth order three point scheme deserves some explanation. Let f_x be v, then the method is to be interpreted as

$$(1 + \frac{1}{6}\delta_x^2) v_{ij} = \frac{1}{2h_x}\bar{\delta}_x f_{ij}$$

or

$$\frac{1}{6}(v_{i+1j} + 4v_{ij} + v_{i-1j}) = \frac{1}{2h_x}\overline{\delta}_x f_{ij}\,.$$

This is an **implicit** formula for the derivative $\dfrac{\partial f}{\partial x}$ at (x_i, y_j). The v_{ij} can be computed from the f_{ij} by solving a tridiagonal system of algebraic equations.

The most common second derivative approximations are

$$f_{xx}|_{ij} = \frac{1}{h_x^2}(f_{ij} - 2f_{i+1j} + f_{i+2j}) + O(h_x)$$

$$f_{xx}|_{ij} = \frac{1}{h_x^2}(f_{ij} - 2f_{i-1j} + f_{i-2j}) + O(h_x)$$

$$f_{xx}|_{ij} = \frac{1}{h_x^2}\delta_x^2 f_{ij} \qquad + O(h_x^2)$$

$$f_{xx}|_{ij} = \frac{1}{h_x^2}\frac{\delta_x^2 f_{ij}}{1 + \frac{1}{12}\delta_x^2} \qquad + O(h_x^4).$$

Remarks:

1. The order of a scheme is given for a uniform mesh.

2. Tables for difference approximations using more than three points and approximations of mixed derivatives are given in Anderson, Tannehill and Pletcher (see [Ref. 14].

3. Difference Representations of PDEs

a. *Truncation Error*

The difference approximations for the derivatives can be expanded in Taylor series. The truncation error (T.E.) is the difference between the partial derivative and its finite difference representation. For example

$$f_x\Big|_{ij} - \frac{1}{h_x}\Delta_x f_{ij} = f_x\Big|_{ij} - \frac{f_{i+1j} - f_{ij}}{h_x} = -f_{xx}\Big|_{ij}\frac{h_x}{2!} - \cdots$$

We use $O(h_x)$ which means that the truncation error satisfies $|T.\ E.| \le K|h_x|$ for $h_x \to 0$, sufficiently small, where K is a positive real constant. Note that $O(h_x)$ does **not** tell us the exact size of the truncation error. If another approximation has a

truncation error of $O(h_x^2)$, we might expect that this would be smaller **only** if the mesh is **sufficiently** fine.

We define the order of a method as the lowest power of the mesh size in the truncation error. Thus Table 1 gives first through fourth order approximations.

The truncation error for a finite difference approximation of a given PDE is defined as the difference between the two. For example, if we approximate the advection equation

$$\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x} = 0\,, \quad c > 0$$

by centered differences

$$\frac{F_{ij+1} - F_{ij-1}}{2\Delta t} + c\frac{F_{i+1j} - F_{i-1j}}{2\Delta x} = 0$$

then the truncation error is

$$T.\ E. \ = \left(\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x}\right)_{ij} - \frac{F_{ij+1} - F_{ij-1}}{2\Delta t} - c\frac{F_{i+1j} - F_{i-1j}}{2\Delta x}$$

$$= -\frac{1}{6}\Delta t^2\frac{\partial^3 F}{\partial t^3} - c\frac{1}{6}\Delta x^2\frac{\partial^3 F}{\partial x^3} - \text{higher powers of } \Delta t \text{ and } \Delta x$$

. We will write

$$T.E. = O(\Delta t^2, \Delta x^2).$$

b. Consistency

A difference equation is said to be consistent or compatible with the partial differential equation when it approaches the latter as the mesh sizes approaches zero. This is equivalent to

$$T.E. \to 0 \quad \text{as mesh sizes} \quad \to 0\,.$$

c. Stability

A numerical scheme is called stable if errors from any source (e. g. truncation, round-off, errors in measurements) are not permitted to grow as the calculation proceeds. Richtmeyer and Morton give a less stringent definition of stability (see [Ref. 15]). A scheme is stable if its solution remains a uniformly bounded function of the initial state for all sufficiently small Δt.

d. Convergence

A scheme is called convergent if the solution to the finite difference equation approaches the exact solution to the PDE with the same initial and boundary conditions as the mesh sizes apporach zero. Lax has proved that under appropriate conditions a consistent scheme is convergent if and only if it is stable.

The Lax Equivalence Theorem states that given a properly posed linear initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence (see [Ref. 15]).

4. Further Examples

Given a PDE and a finite difference mesh one can use any of the following procedures to develop a finite difference scheme.

a. Tables

b. Taylor series expansions

c. polynomial fitting

d. integral methods

e. control volume techniques.

One may get the same scheme by using different approaches. As an example for procedure b we develop a three point second order approximation for $\frac{\partial f}{\partial x}$ on a nonuniform mesh. $\frac{\partial f}{\partial x}$ at point O can be written as a linear combination of values of f at A, O, and B,

$$\left.\frac{\partial f}{\partial x}\right|_O = C_1 f(A) + C_2 f(O) + C_3 f(B)\,.$$

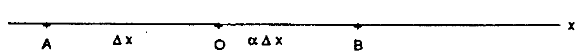

Figure 1. Nonuniform mesh

We use Taylor series to expand $f(A)$ and $f(B)$ about the point O,

$$f(A) = f(O - \Delta x) = f(O) - \Delta x f'(O) + \frac{\Delta x^2}{2} f''(O) - \frac{\Delta x^3}{6} f'''(O) \pm \cdots$$

$$f(B) = f(O + \alpha\Delta x) = f(O) + \alpha\Delta x f'(O) + \frac{\alpha^2\Delta x^2}{2} f''(O) + \frac{\alpha^3\Delta x^3}{6} f'''(O) + \cdots$$

Thus

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_O = (C_1 + C_2 + C_3) f(O) &+ (\alpha C_3 - C_1)\Delta x \left.\frac{\partial f}{\partial x}\right|_O + (C_1 + \alpha^2 C_3)\frac{\Delta x^2}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_O \\ &+ (\alpha^3 C_3 - C_1)\frac{\Delta x^3}{6}\left.\frac{\partial^3 f}{\partial x^3}\right|_O + \cdots \end{aligned}$$

This yields the following system of equations

$$\begin{aligned} C_1 + C_2 + C_3 &= 0 \\ -C_1 \quad + \alpha C_3 &= \frac{1}{\Delta x} \\ C_1 \quad + \alpha^2 C_3 &= 0 \end{aligned}$$

The solution is

$$C_1 = -\frac{\alpha}{(\alpha+1)\Delta x}, \quad C_2 = \frac{\alpha - 1}{\alpha\Delta x}, \quad C_3 = \frac{1}{\alpha(\alpha+1)\Delta x}$$

and thus

$$\frac{\partial f}{\partial x} = \frac{-\alpha^2 f(A) + (\alpha^2 - 1) f(O) + f(B)}{\alpha(\alpha+1)\Delta x} + \frac{\alpha}{6}\Delta x^2 \left.\frac{\partial^3 f}{\partial x^3}\right|_O + \cdots$$

Note that if the grid is uniform then $\alpha = 1$ and this becomes the familiar centered difference.

5. Irregular Mesh

It is more convenient to use a uniform mesh and it is more accurate in some cases. However, in many cases this is not possible due to boundaries which do not coincide with the mesh. In this case several possible cures are given in [Ref. 14]. The most accurate of these is a development of a finite difference approximation which is valid even when the mesh is nonuniform. It can be shown that

$$\left.u_{xx}\right|_O \cong \frac{2}{(1+\alpha)h_x}\left(\frac{u_c - u_O}{\alpha h_x} - \frac{u_O - u_A}{h_x}\right)$$

Similar formula for u_{yy}. Note that for $\alpha = 1$ one obtains the centered difference

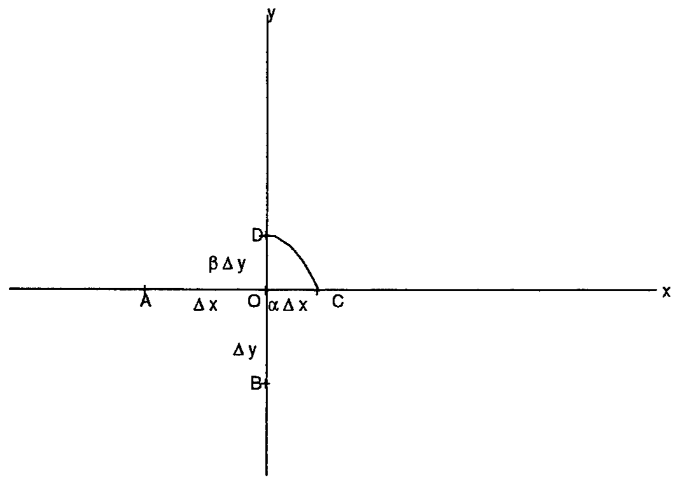

Figure 2. Irregular mesh near curved boundary

approximation.

Due to the need to refine the mesh in some of the domain to maintain the accuracy one is advised to use a coordinate transformation which is covered in Neta (see [Ref. 13]).

6. Stability

The problem of stability is very important in numerical analysis. There are two methods for checking the stability of linear difference equations. The first one is referred to as Fourier or von Neumann and assumes the boundary conditions are periodic. The second one is called the matrix method and takes care of contributions to the error from the boundary.

a. von Neumann analysis

Suppose we solve the advection equation

$$\frac{\partial F}{\partial t} + C\frac{\partial F}{\partial x} = 0\,, \quad c > 0\,,$$

by Lax method (see [Ref. 13])

$$F_{ij+1} = \frac{1}{2}(F_{i+1j} + F_{i-1j}) - c\frac{\Delta t}{2\Delta x}(F_{i+1j} - F_{i-1j})\,.$$

If a term (a single term of Fourier and thus the linearity assumption)

$$F_{ij} = e^{at} e^{ik_m x}$$

is substituted into the difference equation, one obtains the amplification factor

$$e^{at} = \cos\beta - i\nu \sin\beta$$

where

$$\nu = c\frac{\Delta t}{\Delta x} \qquad \text{Courant number}$$

$$\beta = k_m \Delta x \, .$$

The stability requirement is

$$|e^{at}| \leq 1$$

and implies

$$|\nu| \leq 1 \, .$$

This is called the Courant-Friedrichs-Lewy (CFL) condition.

b. Matrix method

Suppose again we solve the advection equation using Lax method but now we assume periodic boundary conditions, i. e.

$$F_{m+1n} = F_{1n}$$

The system of equations obtained is

$$\underline{F}_{n+1} = A\underline{F}_n$$

where

$$\underline{F}_n = \begin{bmatrix} F_{1n} \\ \cdots \\ F_{mn} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \frac{1-\nu}{2} & 0 & \cdots & & \frac{1+\nu}{2} \\ \frac{1+\nu}{2} & 0 & \frac{1-\nu}{2} & & & \\ 0 & & & & & 0 \\ 0 & 0 & \cdots & & 0 & \frac{1-\nu}{2} \\ \frac{1-\nu}{2} & \cdots & 0 & & \frac{1+\nu}{2} & 0 \end{bmatrix}$$

It is clear that the eigenvalues of A are

$$\lambda_j = \cos\frac{2\pi}{m}(j-1) + i\nu \sin\frac{2\pi}{m}(j-1), \quad j = 1, \cdots, m.$$

Clearly the stability of the method depends on

$$|\rho(A)| \leq 1.$$

Note that one obtains the same condition in this case. The two methods yield identical results for periodic boundary condition. It can be shown that this is not the case in general. See work by Hirt (see [Ref. 16]), Warming and Hyett (see [Ref. 17]) and Richtmeyer and Morton (see [Ref. 15]). We will explore the stability of the shallow water equations in more detail using Fourier techniques in the next chapter.

7. Example: Shallow Water Equations in 2D

Arakawa and Lamb ([Ref. 18]) have investigated a finite difference scheme for the nonlinear shallow water equations using square and staggered square grids. We will show two of their examples, one of a square unstaggered grid, and one example of a staggered square grid. For more information on finite difference schemes, refer to their work. [Ref. 18]

a. *Square Grid*

Recalling the linearized version of the shallow water equations

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

Arakawa and Lamb (see [Ref. 18]) use the following finite difference approximation for the unstaggered square grid.

$$\frac{\partial u}{\partial t} - fv + \frac{g}{d}(\overline{\delta_x h})^x = 0$$

$$\frac{\partial v}{\partial t} + fu + \frac{g}{d}(\overline{\delta_y h})^y = 0$$

$$\frac{\partial h}{\partial t} + \frac{H}{d}\left((\overline{\delta_x u})^x + (\overline{\delta_y v})^y\right) = 0$$

where $(\overline{\delta_x h})^x = \frac{1}{2}(h_{i+1,j} - h_{i-1,j})$, in our notation this is $\mu_x \delta_x h$.

b. *Staggered Square Grid*

In simulation, according to Arakawa and Lamb (see [Ref. 18]), a staggered grid approach is best. The reason that a staggered square grid is better for modelling shallow water flow than an unstaggered square grid is that for the same time step, the staggered grid will give higher levels of accuracy. Now let us look at Arakawa and Lamb's example of a staggered square grid. Recalling the shallow water

equations (2.10).

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv &= -g\frac{\partial h}{\partial x}\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu &= -g\frac{\partial h}{\partial y} \qquad (4.1)\\
\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh) &= 0.
\end{aligned}$$

Notice that we substituted h for $(h-h_B)$ therefore ignoring the bottom topography. For the staggered grid, following Arakawa and Lamb's example (see [Ref. 18]), multiply the first of (4.1) by h and add it to the last of (4.1) multiplied by u, also multiply the second of (4.1) by h and add it to the last of (4.1) multiplied by v. This gives us

$$\begin{aligned}
&hu_t+huu_x+hvu_y-fhv+ghh_x+uh_t+u(hu)_x+u(hv)_y=0\\
&hv_t+huv_x+hvv_y+fhu+ghh_y+vh_t+v(hu)_x+v(hv)_y=0.
\end{aligned}$$

This reduces to

$$\begin{aligned}
&(uh)_t+(huu)_x+(hvu)_y-fhv+ghh_x=0\\
&(vh)_t+(huv)_x+(hvv)_y+fhu+ghh_y=0.
\end{aligned}$$

This is another useful form of the momentum equations. Now multiply the first of (4.1) by uh and add it to the last of (4.1) multiplied by $\frac{u^2}{2}$, also multiply the second of (4.1) by vh and add it to the last of (4.1) multiplied by $\frac{v^2}{2}$. This gives us

$$uhu_t+hu^2u_x+huvu_y-fhuv+ghuh_x+\frac{u^2}{2}h_t+\frac{u^2}{2}(hu)_x+\frac{u^2}{2}(hv)_y=0$$

$$vhv_t+hvuv_x+hv^2v_y+fhvu+ghvh_y+\frac{v^2}{2}h_t+\frac{v^2}{2}(hu)_x+\frac{v^2}{2}(hv)_y=0.$$

This now reduces to

$$\begin{aligned}
&\left(h\frac{u^2}{2}\right)_t+\left(hu\frac{u^2}{2}\right)_x+\left(hv\frac{u^2}{2}\right)_y-fhuv+guhh_x=0\\
&\left(h\frac{v^2}{2}\right)_t+\left(hu\frac{v^2}{2}\right)_x+\left(hv\frac{v^2}{2}\right)_y+fhuv+gvhh_y=0.
\end{aligned} \qquad (4.2)$$

This gives the equation for the time change of kinetic energy according to Arakawa and Lamb (see [Ref. 18]). If we multiply the last of (4.1) by gh, we get

$$gh\left[h_t + (hu)_x + (hv)_y\right] = 0$$

or

$$\left(g\frac{h^2}{2}\right)_t + (gh^2u)_x + (gh^2v)_y - gh(uh_x + vh_y) = 0. \tag{4.3}$$

Note that the Coriolis force does not contribute to the change in total kinetic energy. Also note that the sum of terms in (4.2) and (4.3) is zero. It then follows that we have conservation of energy which can then be used in the construction of the finite difference scheme.

Arakawa and Lamb's choice for the differencing of the continuity equation (see [Ref. 18]) was decided in an effort to keep it as simple as possible. At h points, the last equation of (4.1) can be represented as

$$(h_{i,j})_t + \frac{1}{d^2}\left[F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + G_{i,j+\frac{1}{2}} - G_{i,j-\frac{1}{2}}\right] = 0$$

where

$$F_{i+\frac{1}{2},j} \equiv d[h^y u]_{i+\frac{1}{2},j}$$
$$G_{i,j+\frac{1}{2}} \equiv d[h^x v]_{i,j+\frac{1}{2}}$$

are the mass fluxes and are defined at u and v points, respectively. This is semidiscrete therefore the time change terms will be left in differential form.

For the finite difference scheme, the total kinetic energy is conserved during the inertial process. Therefore the terms

$$(uh)_t + (huu)_x + (hvu)_y$$

can be written as follows.

$$\frac{\partial}{\partial t}(H^{(u)}u)_{i,j} + \frac{1}{d^2}\left[\delta_x(\mathcal{F}^{(u)}\bar{u}^x) + \delta_y(\mathcal{G}^{(u)}\bar{u}^y) + \delta_{x'}(\tilde{\mathcal{F}}^{(u)}\bar{u}^{x'}) + \delta_{y'}(\tilde{\mathcal{G}}^{(u)}u^{y'})\right]_{i,j} \tag{4.4}$$

We will use Arakawa and Lamb's reasoning to define $H^{(u)}$, $\mathcal{F}^{(u)}$, $\mathcal{G}^{(u)}$, $\tilde{\mathcal{F}}^{(u)}$, and $\tilde{\mathcal{G}}^{(u)}$ in the next few paragraphs (see [Ref. 18]). (i,j) are used as the indices and are

chosen in such a way that they satisfy

$$\frac{\partial}{\partial t} H_{i,j}^{(u)} + \frac{1}{d^2}\left[\delta_x \mathcal{F}^{(u)} + \delta_y \mathcal{G}^{(u)} + \delta_{x'} \tilde{\mathcal{F}}^{(u)} + \delta_{y'} \tilde{\mathcal{G}}^{(u)}\right]_{i,j} = 0 \tag{4.5}$$

Now multiply (4.5) by u_{ij} and subtract it from (4.4), we will get

$$H_{i,j}^{(u)} \frac{\partial u_{i,j}}{\partial t} + \frac{1}{d^2}\left[\overline{\mathcal{F}^{(u)}\delta_x u^x} + \overline{\mathcal{G}^{(u)}\delta_y u^y} + \overline{\tilde{\mathcal{F}}^{(u)}\delta_x' u^{x'}} + \overline{\tilde{\mathcal{G}}^{(u)}\delta_y' u^{y'}}\right]_{i,j}. \tag{4.6}$$

Take (4.6) and multiply it by u_{ij} and add it to (4.5) which is multipied by $\frac{1}{2}u_{ij}^2$ and a finite difference analog of the first three terms of the first equation of (4.2) is obtained. The finite difference analog is written the same as Arakawa and Lamb wrote it and is as follows (see [Ref. 18]).

$$\begin{aligned}
\tfrac{\partial}{\partial t}(H^{(u)}\tfrac{1}{2}u^2)_{i,j} + \; & \frac{1}{2d^2}\Big[\mathcal{F}_{i+\frac{1}{2},j}^{(u)} u_{i,j}u_{i+1,j} - \mathcal{F}_{i-\frac{1}{2},j}^{(u)} u_{i-1,j}u_{i,j} \\
& + \; \mathcal{G}_{i,j+\frac{1}{2}}^{(u)} u_{i,j}u_{i,j+1} - \mathcal{G}_{i,j-\frac{1}{2}}^{(u)} u_{i,j-1}u_{i,j} \\
& + \; \tilde{\mathcal{F}}_{i+\frac{1}{2},j+\frac{1}{2}}^{(u)} u_{i,j}u_{i+1,j+1} - \tilde{\mathcal{F}}_{i-\frac{1}{2},j-\frac{1}{2}}^{(u)} u_{i-1,j-1}u_{i,j} \\
& + \; \tilde{\mathcal{G}}_{i-\frac{1}{2},j+\frac{1}{2}}^{(u)} u_{i,j}u_{i-1,j+1} - \tilde{\mathcal{G}}_{i+\frac{1}{2},j-\frac{1}{2}}^{(u)} u_{i+1,j-1}u_{i,j}\Big]
\end{aligned}$$

Notice that in this equation, the kinetic energy flux term appears twice, but with the opposite sign the second time. The definition of these terms is not dependent on the form of the equation as long as the total kinetic energy does not increase or decrease over the domain.

The Coriolis term $-fhv$ can be represented at $u_{i,j}$ by

$$-f_j(\overline{h\bar{v}^{y}}^{x})_{i,j}$$

and fhu at $v_{i+\frac{1}{2},j+\frac{1}{2}}$ by

$$(\overline{fh\bar{u}^{x}}^{y})_{i+\frac{1}{2},j+\frac{1}{2}}.$$

The pressure gradient terms ghh_x and ghh_y can be represented as $g[h^x\delta_x h]_{i,j}$ and $g[h^y\delta_y h]_{i+\frac{1}{2},j+\frac{1}{2}}$ respectively. This is a general momentum advection scheme for non-divergent flow to a scheme that will maintain conservation of total energy and also maintain the divergent flow. Horizontal errors due to discretization should be small on the planetary scale due to the large size of the system relative to the grid size. For a more detailed rendition of this example, refer to Arakawa and Lamb (see [Ref. 18]). Neta and Lustman generalized this to nonuniform grid (see [Ref. 19]) .

B. FINITE ELEMENT

In the finite element methods (FEM), the domain is divided into subregions called finite elements, hence the name. The unknown function u is represented as the interpolating polynomial in each element. This representation is continuous along with its derivatives (to a certain order) in each elment.

1. Basic Concepts

One of the ways to formulate the finite element is via the so called weighted residuals method. In this method, the desired function u is replaced by a finite series

$$u^h = \sum_{j=1}^{N} u_j \phi_j.$$

The set of functions ϕ_j are called basis functions. Clearly one can not expect u^h to satisfy the partial differential equation,

$$Lu = f.$$

The residual R is defined as

$$R = Lu^h - f.$$

In order to obtain the undetermined coefficients u_j, one sets the weighted residuals to zero, i.e.

$$\int R w_i = 0 \qquad i = 1, 2, ..., N$$

where the weights w_i may be chosen in various ways. If the weights are chosen to be the same as the basis functions, one obtains Galerkin's method. For collocation, the weights are Dirac delta functions. In the subdomain method one uses the characteristic function of each subdomain as the weight, i.e. $w_i = 1$ in the subdomain Ω_i and zero elsewhere.

Example

Given the equation

$$\frac{d^2u}{dx^2} + u + x = 0 \qquad\qquad 0 < x < 1$$

with homogeneous boundary conditions

$$u(0) = u(1) = 0,$$

we suppose that we can approximate $u(x)$ by

$$u^h = a_1\phi_1(x) + a_2\phi_2(x)$$

and let the basis functions be

$$\begin{aligned} \phi_1 &= x(1-x) \\ \phi_2 &= x^2(1-x). \end{aligned} \tag{4.7}$$

Notice that each basis function satisfies the boundary conditions and therefore u^h satisfies those also. The residual is

$$R = \frac{d^2u^h}{dx^2} + u^h + x = (-2 + x - x^2)a_1 + (2 - 6x + x^2 - x^3)a_2 + x. \tag{4.8}$$

The collocation method yields (using $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}$ as collocation points) the following two equations for the unknowns a_1 and a_2.

$$\begin{aligned} \tfrac{29}{16}a_1 - \tfrac{35}{64}a_2 &= \tfrac{1}{4} \\ \tfrac{7}{4}a_1 + \tfrac{7}{8}a2 &= \tfrac{1}{2} \end{aligned}$$

The approximate solution is then

$$u^h = x(1-x)\frac{42+40x}{217}.$$

The subdomain method yields the following system when the interval is subdivided into two.

$$\begin{aligned}
\int_0^{\frac{1}{2}} R dx &= 0 \\
\int_{\frac{1}{2}}^1 R dx &= 0 \\
-\frac{11}{12}a_1 - \frac{53}{192}a_2 &= -\frac{1}{8} \\
-\frac{11}{12}a_1 - \frac{229}{192}a_2 &= -\frac{3}{8} \\
a_1 &= \frac{97}{517} \\
a_2 &= \frac{88}{517} \\
u^h &= x(1-x)\frac{97+88x}{517}
\end{aligned} \tag{4.9}$$

For Galerkin's method we have the following two integrals to evaluate.

$$\begin{aligned}
\int_0^1 R\phi_1 dx &= 0 \\
\int_0^1 R\phi_2 dx &= 0
\end{aligned} \tag{4.10}$$

with

$$\begin{aligned}
\frac{3}{10}a_1 + \frac{3}{10}a_2 &= \frac{1}{12} \\
\frac{3}{10}a_1 + \frac{13}{105}a_2 &= \frac{1}{20}
\end{aligned}$$

where

$$
\begin{aligned}
a_1 &= \frac{71}{369} \\
a_2 &= \frac{7}{41} \\
u^h &= x(1-x)\left(\frac{71}{369} + \frac{7}{41}x\right)
\end{aligned}
$$

The accuracy of the approximate solution depends on the choice of the basis functions and the method used.

2. Weak Formulation

The fundamental integral statement of the finite element methods can be interpreted as a combination of a weighted residual and a process of integration that reduces the order of continuity required. Going back to the previous example

$$
\begin{aligned}
\frac{d^2u}{dx^2} + u + x &= 0 \\
u(0) = u(1) &= 0 \\
u^h = a_1\phi_1(x) + a_2\phi_2(x) & \\
\int_0^1 \left[\frac{d^2u}{dx^2} + u + x\right]\phi_i dx &= 0 \qquad i = 1, 2
\end{aligned}
\tag{4.11}
$$

Integration by parts yields

$$
\int_1^0 \left(-\frac{du}{dx}\frac{d\phi_i}{dx} + u\phi_i + x\phi_i\right)dx + \frac{du}{dx}\phi_i\Big|_0^1 = 0 \qquad i = 1, 2
$$

It is in this weak form that one substitutes the approximate solution. Notice that $\phi_i(0) = \phi_i(1) = 0$ and thus the boundary term vanishes.

3. Choice of Basis Functions

The accuracy of the methods of weighted residuals depends mainly on the choice of basis functions. In the previous example, we have chosen polynomials. We

now introduce several possible piecewise polynomials. The simplest polynomial is piecewise constants.

a. Piecewise Constant

$$\phi_i(x) = \begin{cases} \alpha & x \epsilon (x_{i-1}, x_{i+1}) \\ \\ 0 & elsewhere \end{cases}$$

b. Piecewise Linear

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1} \\ \\ 0 & elsewhere \end{cases}$$

c. Piecewise Quadratic

For higher order, it is easier to introduce a nondimensional coordinate ξ where $(-1 \leq \xi \leq 1)$. This choice facilitates numerical integration by Gaussian quadratures.

$$\begin{aligned} \phi_{-1}(\xi) &= -\tfrac{1}{2}\xi(1-\xi) \\ \phi_0(\xi) &= 1 - \xi^2 \\ \phi_1(\xi) &= \tfrac{1}{2}\xi(1+\xi) \end{aligned}$$

The subscript indicates the point ξ at which ϕ is one. ϕ is zero at the other two points as shown in the following figure.

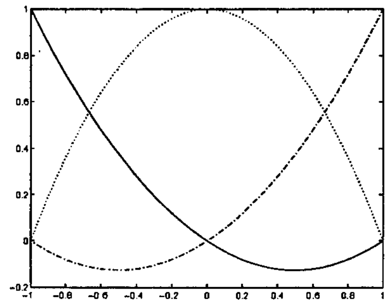

Figure 3. Piecewise quadratic basis functions

d. Piecewise Cubic

It is easy to see that piecewise cubic basis functions are given in the nondimensional coordinate ξ by

$$\phi_{-1}(\xi) = \tfrac{1}{16}(1-\xi)(9\xi^2-1)$$

$$\phi_{\frac{-1}{3}}(\xi) = \tfrac{9}{16}(3\xi-1)(\xi^2-1)$$

$$\phi_{\frac{1}{3}}(\xi) = -\tfrac{9}{16}(3\xi+1)(\xi^2-1)$$

$$\phi_{1}(\xi) = \tfrac{1}{16}(1+\xi)(9\xi^2-1).$$

The above basis functions interpolates along the element using a Lagrange third order polynomial. Another choice is the Hermite polynomial. In addition to continuity of second derivatives over the element, one has first derivative continuity between elements. Therefore Hermite polynomials interpolate the derivatives also. This is useful, for example, in flow problems where one has to differentiate the potential to obtain the velocity field.

Each node is identified with two basis functions.

$$\phi_{01} = \tfrac{1}{4}(1-\xi)^2(\xi+2) \qquad \phi_{01}(-1)=1, \phi_{01}(1)=\phi'_{01}(\pm 1)=0$$

$$\phi_{02} = -\tfrac{1}{4}(1+\xi)^2(\xi-2) \qquad \phi_{02}(1)=1, \phi_{02}(-1)=\phi'_{02}(\pm 1)=0$$

$$\phi'_{11} = \tfrac{1}{4}(1-\xi)^2(\xi+1)$$

$$\phi'_{12} = \tfrac{1}{4}(1+\xi)^2(\xi-1)$$

$$\phi_{ij} = \phi'_{ij}\frac{dt}{d\xi}$$

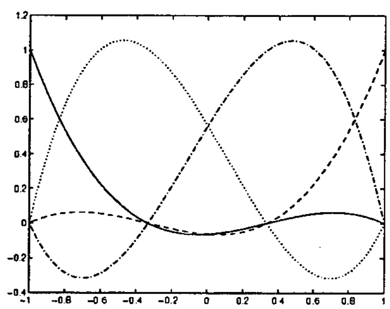

Figure 4. Piecewise cubic basis functions

4. 2D Basis Functions

It is very easy to extend the method of weighted residuals to higher dimensions. In this section we discuss several possibilities of basis functions for rectangular and triangular elements. We close by discussing isoparametric finite elements (irregular quadrilateral elements).

a. *Rectangles*

As in one dimension, one can present Lagrangian and Hermitian basis functions. The Lagrangian basis functions are obtained when a product of two one

dimensional basis functions are used. We will use a quadratic for this example.

$$
\begin{aligned}
\phi_1 &= \tfrac{1}{4}\xi\eta(1-\xi)(1-\eta) && (-1,-1)\\
\phi_2 &= -\tfrac{1}{2}(1-\xi^2)\eta(1-\eta) && (0,-1)\\
\phi_3 &= -\tfrac{1}{4}\xi\eta(1+\xi)(1-\eta) && (1,-1)\\
\phi_4 &= -\tfrac{1}{2}\xi(1-\xi)(1-\eta^2) && (-1,0)\\
\phi_5 &= (1-\xi^2)(1-\eta^2) && (0,0)\\
\phi_6 &= \tfrac{1}{2}\xi(1+\xi)(1-\eta^2) && (1,0)\\
\phi_7 &= -\tfrac{1}{4}\xi\eta(1-\xi)(1+\eta) && (-1,1)\\
\phi_8 &= \tfrac{1}{2}(1-\xi^2)\eta(1+\eta) && (0,1)\\
\phi_9 &= \tfrac{1}{4}\xi\eta(1+\xi)(1+\eta) && (1,1)
\end{aligned}
$$

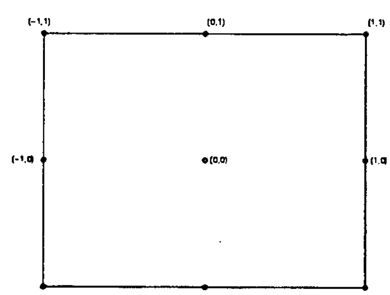

Figure 5. Nodes associated with a rectangular element

The 9 nodes associated with the rectangular element are shown in the figure above. ϕ_i is the basis function having a value of one at the point listed next to its definition and zero at the other 8 nodes.

It is easier to write the basis functions by distinguishing between corner nodes, side nodes, and interior nodes.

Quadratic:

Corner node		$\frac{1}{4}\xi\xi_i(1+\xi\xi_i)\eta\eta_i(1+\eta\eta_i)$
Side node,	$\xi_i = 0$	$\frac{1}{2}\eta\eta_i(1+\eta\eta_i)(1-\xi^2)$
Side node,	$\eta_i = 0$	$\frac{1}{2}\xi\xi_i(1+\xi\xi_i)(1-\eta^2)$
Interior node		$(1-\xi^2)(1-\eta^2)$

Linear:

$$\frac{1}{4}(1+\eta\eta_i)(1+\xi\xi_i)$$

Another possibility, which preserves the properties of the Lagrangian basis functions, but eliminates the interior nodes are called serendipity basis functions. For example, in the case of the quadratic:

Corner node		$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$
Side node,	$\xi_i = 0$	$\frac{1}{2}(1-\xi^2)(1+\eta\eta_i)$
Side node,	$\eta_i = 0$	$\frac{1}{2}(1-\eta^2)(1+\xi\xi_i)$

The cubic and Hermite cubic can be found for example in Lapidus and Pinder [Ref. 20].

b. *Triangles*

The triangular element is the most well known. It allows more accurate presentation of an irregular domain than the rectangular element. A natural representation, called *area coordinates system* was introduced to simplify the integration.

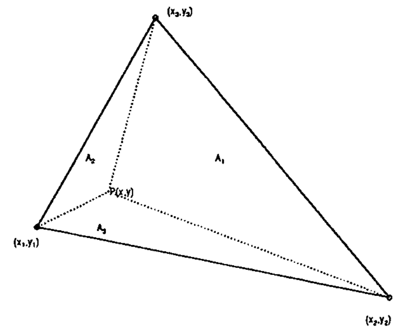

Figure 6. Area coordinates

Each coordinate L_i is defined in terms of the triangle sub area whose base is the line $L_i = 0$ and the vertex is the point $l_i = 1$, thus

$$L_i = \frac{A_i}{A}$$

where A_i are the areas of the subtriangles and A is the area of the whole element. Clearly $A = A_1 + A_2 + A_3$. Also clear is the L_i is unity at node i and zero at the other two nodes. For example $L_i = 1$ if $P(x, y)$ coincides with the vertex (x_1, y_1) and it is zero if P coincides with either of the other two vertices.

Note:

$$L_1 + L_2 + L_3 = 1$$

$$\int_T L_1^{m_1} L_2^{m_2} L_3^{m_3} dxdy = 2A \frac{m_1! m_2! m_3!}{(m_1 + m_2 + m_3 + 2)!}$$

where m_1, m_2, m_3 are nonnegative integers.

$$\int_T \frac{\partial L_i}{\partial y} \frac{\partial L_j}{\partial y} dxdy = \frac{1}{4A}(x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})$$

$$\int_T \frac{\partial L_i}{\partial x} \frac{\partial L_j}{\partial x} dxdy = \frac{1}{4A}(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2})$$

where i and j are cyclic for $1 \leq i, j \leq 3$.

In many cases one requires or prefers numerical integration. We will not discuss this matter, but the reader is refered to Connor and Will (see [Ref. 21]), Brebbia and Connor (see [Ref. 22]) et al.

For quadratic basis functions, one requires three more nodes per element and these are taken as midpoints of each side. The six basis functions in area coordinates are

$$
\begin{aligned}
&4L_2^1 - L_1 \\
&4L_1 L_2 \\
&2L_2^2 - L_2 \\
&4L_2 L_3 \\
&2L_3^2 - L_3 \\
&4L_3 L_1.
\end{aligned}
$$

Several possible higher order triangular elements can be found in Felippa [Ref. 23], Lapidus and Pinder [Ref. 20], and Zienkiewicz [Ref. 24] and others.

c. *Isoparametric*

In the Isoparametric finite elements, all irregularly shaped elements are mapped onto regular elements in order to help with the integration.

Examples

1. Any quadrilateral with straight sides can be mapped onto a square whose vertices are at $(\pm 1, \pm 1)$.

2. Any quadrilateral with curved sides can be mapped onto the same square.

3. Any quadratic triangle with curved sides can be mapped onto an equilateral triangle.

It can be shown the the mapping from the irregular shape in x, y coordinate to ξ, η is given by

$$
\begin{aligned}
x &= \textstyle\sum_{i=1}^{N} x_i \phi_i(\xi, \eta) \\
y &= \textstyle\sum_{i=1}^{N} y_i \phi_i(\xi, \eta)
\end{aligned}
$$

where (x_i, y_i) are the vertices of the quadrilateral and $\phi_i(\xi, \eta)$ are the basis functions of the appropriate order, (in the first case the ϕ_i are linear.)

The evaluation of integrals require the Jacobian of transformation J.

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Isoparametric elements of higher order can be found in Lapidus and Pinder [Ref. 20], Zienkiewicz [Ref. 24] and others.

5. 3D Basis Functions

The extension to three dimensions is straight forward if one has hexahedral elements either Lagrangian or serendidpity (no nodes in the interior). One can also use tetrahedral or pentahedral elements.

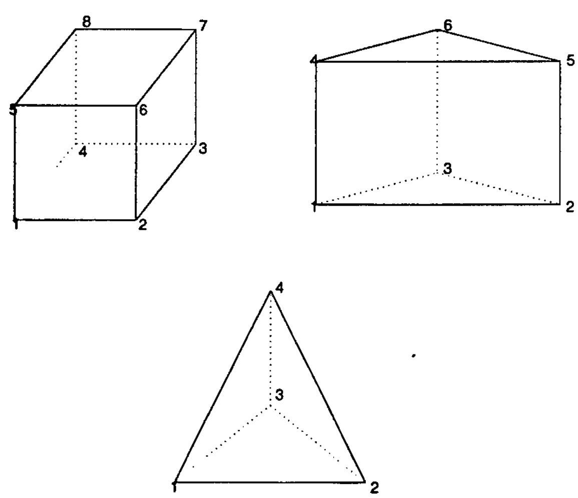

Figure 7. Three Dimensional Elements

Tables for serendipity type can be found in Zienkiewicz [Ref. 24]. Verge [Ref. 25] et al. Area coordinates are replaced by volume coordinates.

6. Example: Shallow Water Equations in 2D

In this section, we give an example of solving the shallow water equation using finite elements. We will ignore the bottom topography. The shallow water equations in vorticity divergence formulation are (as one may recall from Chapter 2)

$$\begin{aligned}
\zeta_t + (uQ)_x + (vQ)_y &= 0 \\
D_t + g\nabla^2 h + \nabla^2 K - (vQ)_x + (uQ)_y &= 0 \\
h_t + [uh]_x + [vh]_y &= 0,
\end{aligned}$$

where h is the geopotential height, u the east/west component of the wind, v the north/south component of the wind and f is still the Coriolis parameter. Hinsman (see [Ref. 26]) shows that the equations can be discretized using finite elements as follows:

$$-\int \left(\bar{\phi}_j \frac{\partial V_j}{\partial x}\frac{\partial V_i}{\partial x} + \bar{\phi}_j \frac{\partial V_j}{\partial y}\frac{\partial V_i}{\partial y} + \frac{\bar{\phi}_j V_j V_i}{\Phi(\Delta t)^2} \right) = \int \left[\frac{\partial}{\partial x}\left((uQ)_j V_j - k_j \frac{\partial V_j}{\partial x} \right) \right] V_i$$

$$-\int \left[\frac{\partial}{\partial y}\left((uQ)_j V_j - k_j \frac{\partial V_j}{\partial y} \right) \right] V_i + \int \frac{1}{\phi \Delta t}\left((\phi u)_j \frac{\partial V_j}{\partial x} + (\phi u)_j \frac{\partial V_j}{\partial y} \right) V_i$$

$$-\int \frac{\phi_j}{\phi(\Delta t)^2}(t-\Delta t) V_j V_i + \int \frac{1}{\Delta t}\left(u_j(t-\Delta t)\frac{\partial V_j}{\partial x} + v_j(t-\Delta t)\frac{\partial V_j}{\partial y} \right) V_i - \oint f_0 u_j V_j V_i \Big|_S^N \tag{4.12}$$

$$-\int \left(\frac{\partial \psi_j}{\partial t}\frac{\partial V_j}{\partial x}\frac{\partial V_i}{\partial x} + \frac{\partial \psi_j}{\partial t}\frac{\partial V_j}{\partial y}\frac{\partial V_i}{\partial y} \right) = -\int \left((uQ)_j \frac{\partial V_j}{\partial x} \right) V_i - \int \left((vQ)_j \frac{\partial V_j}{\partial y} \right) V_i \tag{4.13}$$

$$-\int \left(\frac{\partial \chi_j}{\partial t}\frac{\partial V_j}{\partial x}\frac{\partial V_i}{\partial x} + \frac{\partial \chi_j}{\partial t}\frac{\partial V_j}{\partial y}\frac{\partial V_i}{\partial y} + \frac{\partial \bar{\phi}_j}{\partial t}\frac{\partial V_j}{\partial x}\frac{\partial V_i}{\partial x} + \frac{\partial \bar{\phi}_j}{\partial t}\frac{\partial V_j}{\partial y}\frac{\partial V_i}{\partial y} \right) =$$

$$\int \frac{\partial}{\partial x}\left((vQ)_j V_j - k_j \frac{\partial V_j}{\partial x} \right) V_i - \int \frac{\partial}{\partial y}\left((uQ)_j V_j - k_j \frac{\partial V_j}{\partial y} \right) V_i - \oint f_0 u_j V_j V_i \Big|_S^N \tag{4.14}$$

Divergence $(\nabla^2\chi) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -uf_0$ along the boundary walls. For further look at this example see [Ref. 26]. We have used V_i for the basis functions in order not to confuse it with the geopotential height. Note that (4.13) and (4.14) are time dependent. Also note that Hinsman is using a leap-frog time discretization to obtain each next time step. Leap-frog requires an additional starting value obtained by the Matsuno scheme. (See [Ref. 5].)

V. STABILITY ANALYSIS

Before getting into the stability analysis of the shallow water equations, this paper shall review some basics about integral transforms by focusing on the Fourier transforms. Integral transforms are applied in mathematical type operations by taking an equation which is unsolvable (or really hard to solve) in its original form and transforming it to an equation that is algebraically solvable. Generally, the harder the original problem is to solve, the harder the inverse transform is to find. This chapter will not go into the nitty-gritty of integral transforms, but will offer some insight into why and how they work before applying transforms to the problem at hand. For more information on integral transforms see Miles [Ref. 27]. Simply put, an *integral transform* is of the form $\mathcal{F}(p) = \int_a^b K(p,x)f(x)dx$ where the function to be transformed is $f(x)$, and $K(p,x)$ is the transforming function which is known as the *kernel* of the transform.

A. ONE DIMENSION

1. Fourier Transform

The Fourier Transform in one dimension is

$$\hat{f}(k) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

Also, recall that the inverse Fourier Transform is

$$f(x) = \mathcal{F}^{-1}(\hat{f}(k)) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk.$$

2. Shallow Water Equations

Schoenstadt (see [Ref. 28]) studied the effect of replacing the spatial derivatives in a dispersive wave equation by using Fourier transform techniques. We will repeat some of his work here, but for a more in depth understanding, the reader should consult the original text. Let us consider the effect of semi-discretization on

the solution of a shallow water type model. The shallow water equations in a one-dimensional linearized form with no mean flow, as one may recall from Chapter 3, are

$$\begin{aligned} u_t - fv + gh_x &= 0 \\ v_t + fu &= 0 \\ h_t + Hu_x &= 0. \end{aligned} \tag{5.1}$$

Now if we use the Fourier transform on these equations, (hats ($\hat{}$) symbolize Fourier transforms), we arrive at

$$\begin{aligned} \hat{u}_t &= f\hat{v} - ikg\hat{h} \\ \hat{v}_t &= -f\hat{u} \\ \hat{h}_t &= -ikH\hat{u} \end{aligned}$$

The initial conditions are

$$\begin{aligned} \hat{u}_0 = \hat{u}(k,0) = \textstyle\int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx \\ \hat{v}_0 = \hat{v}(k,0) = \textstyle\int_{-\infty}^{\infty} v(x,0)e^{-ikx}dx \\ \hat{h}_0 = \hat{h}(k,0) = \textstyle\int_{-\infty}^{\infty} h(x,0)e^{-ikx}dx \end{aligned}$$

The transformed shallow water equations are a coupled set of constant coefficient ordinary differential equations which can be solved fairly easily (see [Ref. 28]) and are

$$\begin{aligned} \hat{u} &= \frac{\alpha_2}{\nu}e^{i\nu t} - \frac{\alpha_3}{\nu}e^{-i\nu t} \\ \hat{v} &= ikg\alpha_1 + \frac{if\alpha_2}{\nu^2}e^{i\nu t} + \frac{if\alpha_3}{\nu^2}e^{-i\nu t} \\ \hat{h} &= f\alpha_1 - \frac{kH\alpha_2}{\nu^2}e^{i\nu t} + \frac{kH\alpha_3}{\nu^2}e^{-i\nu t} \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \nu^2 &= f^2 + k^2gH = f^2(1+\lambda^2k^2) \\ \lambda &= \frac{\sqrt{gH}}{f}. \end{aligned}$$

The α_i's are picked to satisfy the initial conditions. Recalling that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ we can rewrite (5.2) by using the intial conditions to solve for

α_i and then simplifying to get

$$\begin{aligned}
\hat{u}(k,t) &= \hat{u}_0 \cos \nu t + \frac{f \hat{v}_0}{\nu} \sin \nu t - \frac{ikg\hat{h}_0}{\nu} \sin \nu t \\
\hat{v}(k,t) &= -\frac{f\hat{u}_0}{\nu} \sin \nu t + \left\{ \frac{k^2 H f}{\nu^2} + \frac{f^2}{\nu^2} \cos \nu t \right\} \hat{v}_0 + \frac{ikgf}{\nu^2} \left\{ 1 - \cos \nu t \right\} \hat{h}_0 \\
\hat{h}(k,t) &= -\frac{ikH\hat{u}_0}{\nu} \sin \nu t - \frac{ikHf}{\nu^2} \left\{ 1 - \cos \nu t \right\} \hat{v}_0 + \left\{ \frac{f^2}{\nu^2} + \frac{k^2 H f}{\nu^2} \cos \nu t \right\} \hat{h}_0.
\end{aligned}$$

The steady state solutions are

$$\begin{aligned}
\hat{u}_s(k) &= 0 \\
\hat{v}_s(k) &= \frac{k^2 g H}{\nu^2} \hat{v}_0 + \frac{ikgf}{\nu^2} \hat{h}_0 \\
\hat{h}_s(k) &= -\frac{ikHf}{\nu^2} \hat{v}_0 - \frac{f^2}{\nu^2} \hat{h}_0
\end{aligned}$$

or to rewrite

$$\begin{aligned}
\hat{u}_s(k) &= 0 \\
\hat{v}_s(k) &= \hat{v}_0 + \frac{f^2}{\nu^2} \left(\frac{ikg}{f} \hat{h}_0 - \hat{v}_0 \right) \\
\hat{h}_s(k) &= \hat{h}_0 + \frac{H}{f} \frac{f^2}{\nu^2} ik \left(\frac{ikg}{f} \hat{h}_0 - \hat{v}_0 \right).
\end{aligned}$$

By noting that

$$\int_{-\infty}^{\infty} e^{\frac{-|x|}{\lambda}} e^{-ikx} dx = \frac{2\lambda}{1 + k^2 \lambda^2} = \frac{2\lambda f^2}{\nu^2}$$

and using the convolution theorem, Schoenstadt (see [Ref. 28]) inverted the steady state transformed equations to yield

$$u_s(x) = 0$$

$$v_s(x) = v(x,0) + \frac{1}{2\lambda}\int_{-\infty}^{\infty} e^{\frac{-|x-s|}{\lambda}} \left(\frac{g}{f}\frac{\partial h}{\partial x}(s,0) - v(s,0)\right) ds$$

$$h_s(x) = h(x,0) - \frac{H}{2\lambda^2 f}\int_{-\infty}^{\infty} \operatorname{sgn}(x-s) e^{\frac{-|x-s|}{\lambda}} \left(\frac{g}{f}\frac{\partial h}{\partial x}(s,0) - v(s,0)\right) ds.$$

Note that in the one dimensional case, $\hat{u}(k,0)$ does not contribute to the steady state solutions. Schoenstadt (see [Ref. 28]) goes on to explore the transient parts of the model, and these can be studied in more detail in his 1977 paper.

Schoenstadt (see [Ref. 29]) also analyzed a difference scheme for one dimension shallow water equations that is similar to the one we will present later in this chapter for two dimensions.

B. TWO DIMENSIONS

1. Fourier Transform

Recall that the Fourier Transform in 2 dimensions is

$$\hat{f}(w_1, w_2) = \mathcal{F}(f(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-i(w_1 x + w_2 y)} dx dy,$$

and the inverse Fourier Transform is

$$f(x,y) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(w_1, w_2) e^{i(w_1 x + w_2 y)} dw_1 dw_2.$$

Also recall several elementary properties such as:

$$\begin{aligned}
\mathcal{F}(f_t) &= \frac{\partial}{\partial t}\mathcal{F}(f) \\
\mathcal{F}(f_x) &= i w_1 \mathcal{F}(f) \\
\mathcal{F}(f_y) &= i w_2 \mathcal{F}(f) \\
\mathcal{F}(\nabla f) &= i \vec{w} \mathcal{F}(f) \\
\mathcal{F}(\nabla^2 f) &= -\vec{w}^2 \mathcal{F}(f)
\end{aligned}$$

We will use several of these properties later in this chapter when dealing with the continuous case and also the semi-discrete case. The convolution theorem will also be used in this section and, in two dimensions, it is

$$f(x,y) = \mathcal{F}^{-1}(\hat{g}(k,l) \cdot \hat{h}(k.l)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_0,y_0)h(x-x_0,y-y_0)dx_0dy_0.$$

2. Continuous Case

Recall the linearized shallow water equations from Chapter 3.

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0 \tag{5.3}$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Following the work of Neta (see [Ref. 2]), we will go through the reasoning done by Neta and by Neta and DeVito (see [Ref. 7]) to arrive at their solution. Let us take the first equation of (5.1) and transform it using Fourier,

$$\mathcal{F}\left(\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x}\right)e^{-i(kx+ly)}dxdy$$

$$= \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ue^{-i(kx+ly)}dxdy - f\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ve^{-i(kx+ly)}dxdy + g\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial h}{\partial x}e^{-i(kx+ly)}dxdy$$

$$= \frac{\partial \hat{u}}{\partial t} - f\hat{v} + igk\hat{h} = 0.$$

Similarly

$$\mathcal{F}\left(\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y}\right) = \frac{\partial \hat{v}}{\partial t} + f\hat{u} + igl\hat{h} = 0.$$

$$\mathcal{F}\left(\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) = \frac{\partial \hat{h}}{\partial t} + iHk\hat{u} + iHl\hat{v} = 0.$$

This is the reasoning that Neta and De Vito (see [Ref. 7]) used to show that the Fourier transform of (5.1) is

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} \hat{u}_s \\ \hat{v}_s \\ \hat{h}_s \end{bmatrix} + \frac{1}{\nu^2} A \begin{bmatrix} \hat{u}_0 \\ \hat{v}_0 \\ \hat{h}_0 \end{bmatrix} \cos \nu t + \frac{1}{\nu^2} B \begin{bmatrix} \hat{u}_0 \\ \hat{v}_0 \\ \hat{h}_0 \end{bmatrix} \sin \nu t \tag{5.4}$$

where A and B are given by

$$A = \begin{bmatrix} f^2 + ghk^2 & gHkl & igfl \\ gHkl & f^2 + ghl^2 & -igfk \\ -iHfl & iHfk & gH(k^2 + l^2) \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & f & -igk \\ -f & 0 & -igl \\ -iHk & -iHl & 0 \end{bmatrix}.$$

Note that if you reduce the terms to one dimension, the results are the same as shown by Schoenstadt in the one dimensional case. The steady state solution for (5.4) is

$$\begin{bmatrix} \hat{u}_s \\ \hat{v}_s \\ \hat{h}_s \end{bmatrix} = \frac{1}{\nu^2} C \begin{bmatrix} \hat{u}_0 \\ \hat{v}_0 \\ \hat{h}_0 \end{bmatrix}$$

where

$$C = \begin{bmatrix} ghl^2 & -gHkl & -igfl \\ -gHkl & ghk^2 & igfk \\ iHfl & -iHfk & f^2 \end{bmatrix} \tag{5.5}$$

$\hat{u}_0$, $\hat{v}_0$ and $\hat{h}_0$ are the Fourier transforms of the initial conditions. The frequency ν is given by $\nu = f\sqrt{1 + \lambda^2(k^2 + l^2)}$ where λ is the Rossby radius of deformation (i.e. $\lambda = \frac{\sqrt{gH}}{f}$).

Following Neta (see [Ref. 30]), we shall now take the inverse Fourier transform of (5.5) to obtain the steady state solution. We will follow the arrangement of the

variables used there. They are

$$\begin{aligned}
\hat{D} &= ik\hat{u}_0 + il\hat{v}_0 \\
\hat{\zeta} &= il\hat{u}_0 - ik\hat{v}_0 \\
\hat{d}_x &= i\frac{g}{f}k\hat{h}_0 - \hat{v}_0 \\
\hat{d}_y &= i\frac{g}{f}l\hat{h}_0 + \hat{u}_0
\end{aligned}$$

where the steady state solution is

$$\begin{aligned}
\hat{u}_s &= \hat{u}_0 + \tfrac{f^2}{\nu^2}(\lambda^2 ik\hat{D} - \hat{d}_y) \\
\hat{v}_s &= \hat{v}_0 + \frac{f^2}{\nu^2}(\lambda^2 il\hat{D} + \hat{d}_x) \\
\hat{h}_s &= \hat{h}_0 + \frac{H}{f}\frac{f^2}{\nu^2}(ik\hat{d}_x + il\hat{d}_y)
\end{aligned}$$

We can now invert these equations using the convolution theorem and the following integrals.

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{ir\rho\cos(\phi-\theta)}d\phi &= J_0(r\rho) \\
\frac{1}{2\pi}\int_0^{\infty} \frac{J_0(r\rho)}{1+\lambda^2\rho^2}\rho d\rho &= \frac{1}{2\pi\lambda^2}K_0\left(\frac{r}{\lambda}\right) \qquad (5.6) \\
\mathcal{F}^{-1}\left(\frac{f^2}{\nu^2}\right) &= \frac{1}{2\pi\lambda^2}K_0\left(\frac{\sqrt{x^2+y^2}}{\lambda}\right)
\end{aligned}$$

J_0 and K_0 are Bessel functions of order zero. Using these integrals, the inverse obtained from them, and the convolution theorem, we have the steady state solutions.

$$
\begin{aligned}
u_s(x,y) &= u_0(x,y) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}K_0(q(\sigma,\tau))D(\sigma,\tau,0)d\sigma d\tau \\
&\quad -\frac{1}{2\pi\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_0(q(\sigma,\tau))d_y(\sigma,\tau,0)d\sigma d\tau, \\
v_s(x,y) &= v_0(x,y) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}K_0(q(\sigma,\tau))D(\sigma,\tau,0)d\sigma d\tau \\
&\quad +\frac{1}{2\pi\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_0(q(\sigma,\tau))d_x(\sigma,\tau,0)d\sigma d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
h_s(x,y) &= h_0(x,y) - \frac{H}{f}\frac{1}{2\pi\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial y}K_0(q(\sigma,\tau))d_y(\sigma,\tau,0)\right. \\
&\quad \left.+\frac{\partial}{\partial x}K_0(q(\sigma,\tau))d_x(\sigma,\tau,0)\right)d\sigma d\tau
\end{aligned}
$$

where

$$
\begin{aligned}
q(\sigma,\tau) &= \frac{\sqrt{(x-\sigma)^2+(y-\tau)^2}}{\lambda} \\
D(\sigma,\tau,0) &= \frac{\partial u_0}{\partial \sigma} + \frac{\partial v_0}{\partial \tau}. \\
d_x(\sigma,\tau,0) &= \frac{g}{f}\frac{\partial h_0}{\partial \tau} - v_0 \\
d_y(\sigma,\tau,0) &= \frac{g}{f}\frac{\partial h_0}{\partial \sigma} + u_0.
\end{aligned}
$$

3. Semi-discrete Equations

Here we will present the semi-discrete shallow water system and then show the inverse Fourier transform for certain selections of α, β, and γ's. Semi-discrete means that time is **not** discretized. In the semi-discrete case, the shallow water equations

are

$$\alpha\frac{\partial \hat{u}}{\partial t} - \beta f\hat{v} + ig\gamma_1\hat{h} = 0$$

$$\alpha\frac{\partial \hat{v}}{\partial t} + \beta f\hat{u} + ig\gamma_2\hat{h} = 0 \qquad (5.7)$$

$$\alpha\frac{\partial \hat{h}}{\partial t} + iH\left(\gamma_1\hat{u} + \gamma_2\hat{v}\right) = 0$$

where α, β, γ_1, and γ_2 depend on the scheme used (see tables II and III) from Neta [Ref. 30].

Table II. Filter weights for second and fourth order finite differences

Scheme	α	β	γ_1	γ_2
A2	1	1	$\frac{\sin kd}{d}$	$\frac{\sin ld}{d}$
B2	1	1	$\frac{\sin \frac{kd}{2}\cos\frac{ld}{2}}{\frac{d}{2}}$	$\frac{\sin \frac{ld}{2}\cos\frac{kd}{2}}{\frac{d}{2}}$
C2	1	$\cos\frac{kd}{2}\cos\frac{ld}{2}$	$\frac{\sin \frac{kd}{2}}{\frac{d}{2}}$	$\frac{\sin \frac{ld}{2}}{\frac{d}{2}}$
D2	1	$\cos\frac{kd}{2}\cos\frac{ld}{2}$	$\frac{\sin kd\cos\frac{ld}{2}}{d}$	$\frac{\sin ld\cos\frac{kd}{2}}{d}$
A4	1	1	$\frac{8\sin kd - \sin 2kd}{6d}$	$\frac{8\sin ld - \sin 2ld}{6d}$
B4	1	1	$\frac{(-\sin\frac{3kd}{2} + 27\sin\frac{kd}{2})\cos\frac{ld}{2}}{12d}$	$\frac{(-\sin\frac{3ld}{2} + 27\sin\frac{ld}{2})\cos\frac{kd}{2}}{12d}$
C4	1	$\cos\frac{kd}{2}\cos\frac{ld}{2}$	$\frac{-\sin\frac{3kd}{2} + 27\sin\frac{kd}{2}}{12d}$	$\frac{-\sin\frac{3ld}{2} + 27\sin\frac{ld}{2}}{12d}$
D4	1	$\cos\frac{kd}{2}\cos\frac{ld}{2}$	$\frac{(8\sin kd - \sin 2kd)\cos\frac{ld}{2}}{6d}$	$\frac{(8\sin ld - \sin 2ld)\cos\frac{kd}{2}}{6d}$

Table III. Filter weights for finite elements

Scheme	$\alpha = \beta$	γ_1	γ_2
FET	$\dfrac{3+\cos kd + 2\cos\frac{kd}{2}\cos ld}{6}$	$\dfrac{2(\sin kd + \sin\frac{kd}{2})\cos ld}{3d}$	$\dfrac{\cos\frac{kd}{2}\sin ld}{d}$
FER	$\dfrac{(2+\cos kd)(2+\cos ld)}{9}$	$\dfrac{(2+\cos ld)(\sin kd)}{3d}$	$\dfrac{(2+\cos kd)(\sin ld)}{3d}$

The two schemes for finite elements in Table III are for triangles (Finite Element-Triangle or FET) and rectangles (FER).

The Fourier transform of the steady state solution is similar to the steady state solution of the continuous case and is written as follows with obvious changes from the continuous case.

$$\begin{bmatrix} \hat{u}_s \\ \hat{v}_s \\ \hat{h}_s \end{bmatrix} = \frac{1}{\alpha^2\nu^2} C_D \begin{bmatrix} \hat{u}_0 \\ \hat{v}_0 \\ \hat{h}_0 \end{bmatrix}$$

where

$$C_D = \begin{bmatrix} gh\gamma_2^2 & -gH\gamma_1\gamma_2 & -igf\beta\gamma_2 \\ -gH\gamma_1\gamma_2 & gh\gamma_1^2 & igf\beta\gamma_1 \\ iHf\beta\gamma_2 & -iHf\beta\gamma_1 & \beta^2 f^2 \end{bmatrix} \tag{5.8}$$

and

$$\begin{aligned} \alpha\nu &= \beta f\sqrt{1+\lambda_D^2(\gamma_1^2+\gamma_2^2)} \\ \lambda_D &= \frac{\sqrt{gH}}{\beta f}. \end{aligned}$$

There is a great deal of similarity between the continuous case and the discrete case up to this point. We can carry it further and define as before

$$\begin{aligned} \hat{D}(k,l,0) &= i\gamma_1\hat{u}_0 + i\gamma_2\hat{v}_0 \\ \hat{\zeta}(k,l,0) &= i\gamma_2\hat{u}_0 - i\gamma_1\hat{v}_0 \\ \hat{d}_x(k,l,0) &= i\frac{g}{f}\gamma_1\hat{h}_0 - \beta\hat{v}_0 \\ \hat{d}_y(k,l,0) &= i\frac{g}{f}\gamma_2\hat{h}_0 + \beta\hat{u}_0. \end{aligned}$$

Therefore the steady state Fourier transforms of the semi-discrete shallow water equations are

$$\hat{u}_s = \hat{u}_0 + \frac{f^2}{\alpha^2\nu^2}(\lambda^2 i\gamma_1 \hat{D} - \beta\hat{d}_y)$$

$$\hat{v}_s = \hat{v}_0 + \frac{f^2}{\alpha^2\nu^2}(\lambda^2 i\gamma_2 \hat{D} - \beta\hat{d}_x) \tag{5.9}$$

$$\hat{h}_s = \hat{h}_0 + \frac{H}{f}\frac{f^2}{\alpha^2\nu^2}(i\gamma_1\hat{d}_x + i\gamma_2\hat{d}_y).$$

Notice that the Rossby radius of deformation (λ) was used instead of the its discrete analog (λ_D).

a. *Finding the Steady State Solutions for Scheme A2*

Let

$$I_m(x,y) = \mathcal{F}^{-1}\left(\frac{f^2}{\alpha^2\nu^2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(kx+ly)}}{\beta^2(k,l)+\lambda^2[\gamma_1^2(k,l)+\gamma_2^2(k,l)]}dkdl.$$

For scheme A2, $\mathcal{F}^{-1}(i\gamma_1\hat{q}(k,l))$ is

$$\begin{aligned}
\mathcal{F}^{-1}(\frac{\sin kd}{kd}ik\hat{q}(k,l)) &= \int_{-\infty}^{\infty}\int_{-d}^{d}\frac{\partial q(x_0,y_0)}{\partial x}\frac{1}{2d}\delta(y-y_0)dx_0dy_0 \\
&= \frac{1}{2d}\int_{-d}^{d}\frac{\partial q(x_0,y)}{\partial x}dx_0 \\
&= \frac{1}{2d}[q(d,y)-q(-d,y)]
\end{aligned}$$

and $\mathcal{F}^{-1}(i\gamma_2\hat{q}(k,l))$ is

$$\begin{aligned}
\mathcal{F}^{-1}(\frac{\sin ld}{ld}il\hat{q}(k,l)) &= \int_{-\infty}^{\infty}\int_{-d}^{d}\frac{\partial q(x_0,y_0)}{\partial y}\frac{1}{2d}\delta(x-x_0)dy_0dx_0 \\
&= \frac{1}{2d}\int_{-d}^{d}\frac{\partial q(x,y_0)}{\partial y}dy_0 \\
&= \frac{1}{2d}[q(x,d)-q(x,-d)].
\end{aligned}$$

It follows in the semi-discrete case of the shallow water equations, by using the convolution theorem the steady state solution for scheme A2 is

$$u_s(x,y) = u_0(x,y) + \lambda^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \frac{\partial I_m(x-\sigma, y-\tau)}{\partial x}\frac{1}{2d}[u_0(d,\tau,0) - u_0(-d,\tau,0)] \right.$$

$$\left. -I_m(x-\sigma,y-\tau)\left(\frac{1}{2d}\right)^2 [\{v_0(d,d) - v_0(d,-d)\} - \{v_0(-d,d) - v_0(-d,-d)\}] \right\} d\sigma d\tau$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_m(x-\sigma,y-\tau)d_y(\sigma,\tau,0)d\sigma d\tau$$

$$v_s(x,y) = v_0(x,y) + \lambda^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \frac{\partial I_m(x-\sigma, y-\tau)}{\partial y}\frac{1}{2d}[v_0(\sigma,d,0) - v_0(\sigma,-d,0)] \right.$$

$$\left. -I_m(x-\sigma,y-\tau)\left(\frac{1}{2d}\right)^2 [\{u_0(d,d) - u_0(d,-d)\} - \{u_0(-d,d) - u_0(-d,-d)\}] \right\} d\sigma d\tau$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_m(x-\sigma,y-\tau)d_x(\sigma,\tau,0)d\sigma d\tau$$

$$h_s(x,y) = h_0(x,y) + \frac{H}{f}\frac{1}{2d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \frac{\partial I_m(x-\sigma, y-\tau)}{\partial x}\frac{g}{f}[h_0(d,\tau,0) - h_0(-d,\tau,0)] \right.$$

$$-I_m(x-\sigma,y-\tau)[v_0(d,\tau,0) - v_0(-d,\tau,0)]$$

$$-I_m(x-\sigma,y-\tau)[u_0(\sigma,d,0) - u_0(\sigma,-d,0)]$$

$$\left. +\frac{\partial I_m(x-\sigma,y-\tau)}{\partial y}\frac{g}{f}[h_0(\sigma,d,0) - h_0(\sigma,-d,0)] \right\} d\sigma d\tau \tag{5.10}$$

where

$$d_x(x,y,0) = \frac{g}{f}\frac{1}{2d}\left[h_0(d,y) - h_0(-d,y)\right] - v_0$$

$$d_y(x,y,0) = \frac{g}{f}\frac{1}{2d}\left[h_0(x,d) - h_0(x,-d)\right] + u_0.$$

Note that $d_x(x,y,0)$ is a centered difference approximation which approaches u_x as d approaches 0. One should be able to obtain similar results for the other schemes listed in the tables.

b. Solving for I_m

The reason for this thesis is to show which semi-discrete method has an inverse Fourier transform which can be back transformed in a closed form. By the Lax equivalence theorem, if we can show that I_m is bounded in some sense, then the solution is stable also. Numerical analysis may be used to solve this problem, but a simple closed formula is considered a more ideal solution. Let us look again at

$$I_m(x,y) = \mathcal{F}^{-1}\left(\frac{f^2}{\alpha^2\nu^2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(kx+ly)}}{\beta^2(k,l)+\lambda[\gamma_1^2(k,l)+\gamma_2^2(k,l)]}dkdl. \qquad (5.11)$$

Note that the limit of I_m for any of the schemes when $d \to 0$ is

$$I_m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(kx+ly)}}{1+\lambda^2\left(k^2+l^2\right)}dkdl.$$

By using the identity $\frac{1}{a} = \int_0^\infty e^{-\lambda a}d\lambda$, we have the following triple integral,

$$I_m = \int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kx+ly)}e^{-\left\{\mu+\mu\lambda^2\left(k^2+l^2\right)\right\}}dkdld\mu,$$

which can be rewritten as

$$I_m = \int_0^\infty e^{-\mu}\left(\int_{-\infty}^{\infty} e^{ikx}e^{-\mu\lambda^2k^2}dk\int_{-\infty}^{\infty} e^{ily}e^{-\mu\lambda^2l^2}dl\right)d\mu.$$

Evaluate the three integrals separately in this form and it easily can be shown that

$$\lim_{d\to 0}\lambda^2 I_m = \frac{1}{2\pi}K_0$$

where K_0 is the Bessel function of order zero from (5.6). This shows that the steady state solution of the semi-discrete system (5.10) approaches the corresponding solution (5.5) of the continuous case.

Refering to Neta and Devito (see [Ref. 7]) we have several second and fourth order finite difference equations (Table II). In scheme A2 we have

$$I_{A2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(kx+ly)}}{1+\lambda^2\left(\frac{\sin^2(kd)}{d^2}+\frac{\sin^2(ld)}{d^2}\right)}dkdl. \qquad (5.12)$$

By noting in (5.12) that we have, for example, $\cos k$ in the numerator and that the denominator is positive and finite, we see that I_{A2} does not converge. What one can do is look at I_{A2} to determine if there is any useful information which can be drawn from it. This information then can be used to predict some aspect of the semi-discrete shallow water equations. Latta ([Ref. 31]) suggested the following reasoning. One can rewrite I_{A2} as

$$I_{A2} = \frac{1}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(kx+ly)}}{\frac{1}{C}+\left(\sin^2(kd)+\sin^2(ld)\right)}dkdl$$

where $C = \dfrac{\lambda^2}{d^2}$. Notice that C is a constant. Now if one lets $kd \to k$ and $ld \to l$, I_{A2} becomes

$$I_{A2} = \frac{1}{Cd^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(k\frac{x}{d}+l\frac{y}{d})}}{\frac{1}{C}+\left(\sin^2(k)+\sin^2(l)\right)}dkdl. \tag{5.13}$$

Let $k \to k+\pi$ and $l \to l+\pi$. This shows that I_{A2} is infinite where $I_{A2} = e^{i\frac{x}{d}\pi}I_{A2}$ and $I_{A2} = e^{i\frac{y}{d}\pi}I_{A2}$. Therefore I_{A2} is infinite when $x = 2md$ and $y = 2nd$ where m and n are integers and $I_{A2} = 0$ otherwise. The solution to I_{A2} will look like

$$Cd^2 I_{A2} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_{mn}\delta(k-2m)\delta(l-2n) = \mathcal{F}\left(\frac{1}{\frac{1}{C}+\sin^2(k)+\sin^2(l)}\right).$$

Now take the inverse Fourier Transform of the middle term to get

$$\left(\frac{1}{2\pi}\right)^2\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_{mn}e^{i2mk}e^{i2nl} = \frac{1}{\frac{1}{C}+\sin^2(k)+\sin^2(l)}$$

By noting that we are only interested in the real part of I_{A2}, we can write the equation as

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_{mn}\cos(2mk)\cos(2nl) = \frac{4\pi^2}{\frac{1}{C}+\sin^2(k)+\sin^2(l)}.$$

Using the Fourier Cosine series, we can write the coefficients α_{mn} as

$$\alpha_{mn} = \int_0^{2\pi}\int_0^{2\pi}\frac{\cos(2mk)\cos(2nl)}{\frac{1}{C}+\sin^2(k)+\sin^2(l)}dkdl.$$

Now using the trigonometric substitution $\sin^2\beta = \dfrac{1-\cos 2\beta}{2}$, we get

$$\alpha_{mn} = \int_0^{2\pi}\int_0^{2\pi}\frac{\cos(2mk)\cos(2nl)}{\dfrac{1}{C}+1-\dfrac{\cos(2k)+\cos(2l)}{2}}dkdl$$

which can be written as

$$\alpha_{mn} = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos(2mk)\cos(2nl)}{\frac{1+C}{C}\left(1 - \frac{\cos(2k)+\cos(2l)}{2\frac{1+C}{C}}\right)} dkdl.$$

Using the geometric series expansion $(\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...)$, we can write α_{mn} as

$$\alpha_{mn} = \int_0^{2\pi}\int_0^{2\pi} \cos(2mk)\cos(2nl)\sum_{\nu=0}^{\infty}\left[\frac{C}{2+2C}(\cos(2k)+\cos(2l))\right]^{\nu} dkdl \qquad (5.14)$$

Note that $\cos(2mk)\cos(2nl) \leq 1$ and $(\cos(2k)+\cos(2l))^{\nu} \leq 2^{\nu}$. Therefore

$$\alpha_{mn} \leq 4\pi^2 \sum_{\nu=0}^{\infty}\left(\frac{C}{1+C}\right)^{\nu}.$$

This means that α_{mn} is bounded and it follows that although I_{A2} infinite, it is stable at each lattice point.

Since $(a+b)^{\nu} = \sum_{r=0}^{\nu}\begin{pmatrix}\nu\\ r\end{pmatrix} a^{\nu-r}b^{r}$, (5.4) becomes

$$\alpha_{mn} = \sum_{\nu=0}^{\infty}\left(\frac{C}{2+2C}\right)^{\nu}\sum_{r=0}^{\nu}\begin{pmatrix}\nu\\ r\end{pmatrix}\int_0^{2\pi}\cos(2km)\cos^{r}(2k)dk\int_0^{2\pi}\cos(2ln)\cos^{\nu-r}(2l)dl.$$

Using Gradshteyn and Ryshik (see [Ref. 32], p. 374), one gets the following definite integral.

$\int_0^{\pi}\cos(2mk)cos^{r}(2k)dk =$

$$(1+(-1)^{2m+r})\begin{cases} s\dfrac{r!}{(2m-r)(2m-r+2)...(2m+r)} & [r < 2m]; \\ \dfrac{\pi}{2^{r+1}}\begin{pmatrix} r\\ i\end{pmatrix} & [2m < r \text{ and } r-2m = 2i]; \\ \dfrac{r!}{(2i+1)!!(4m+2i+1)!!} & [2m < r \text{ and } r-2m = 2i+1] \end{cases} \qquad (5.15)$$

where

$$s = \begin{cases} 0 & [2m - r = 2i]; \\ 1 & [2m - r = 4i + 1]; \\ -1 & [2m - r = 4i - 1] \end{cases}$$

Since α_{mn} is the same across any 2π interval, we can write it as

$$\alpha_{mn} = \sum_{\nu=0}^{\infty} \left(\frac{C}{2+2C}\right)^{\nu} \sum_{r=0}^{\nu} \begin{pmatrix} \nu \\ r \end{pmatrix} \int_{-\pi}^{\pi} \cos(2km) \cos^{r}(2k) dk \int_{-\pi}^{\pi} \cos(2ln) \cos^{\nu-r}(2l) dl.$$

Noting that cosine is even, this can be rewritten as

$$\alpha_{mn} = 4\sum_{\nu=0}^{\infty} \left(\frac{C}{2+2C}\right)^{\nu} \sum_{r=0}^{\nu} \begin{pmatrix} \nu \\ r \end{pmatrix} \int_{0}^{\pi} \cos(2km) \cos^{r}(2k) dk \int_{0}^{\pi} \cos(2ln) \cos^{\nu-r}(2l) dl.$$

By using (5.15), we note that if ν is odd, then $\alpha_{mn} = 0$. Also if ν is even and r is odd then $\alpha_{mn} = 0$. Therefore $\nu = (2m + 2n + 2i + 2j)$ where $r = 2m + 2i$. Now we note if $r < 2m$ or $\nu - r < 2n$, then $\alpha_{mn} = 0$. From these observations of (5.15), it follows that

$$\alpha_{mn} = 4\sum_{\nu=0}^{\infty} \left(\frac{C}{2+2C}\right)^{\nu} \sum_{r=0}^{\nu} \begin{pmatrix} \nu \\ r \end{pmatrix} \frac{\pi}{2^{r+1}} \begin{pmatrix} r \\ i \end{pmatrix} \frac{\pi}{2^{\nu-r+1}} \begin{pmatrix} \nu - r \\ j \end{pmatrix}.$$

This can be reduced to

$$\pi^2 \sum_{\nu=2m+2n}^{\infty}{}' \sum_{r=2m}^{\nu}{}' \left(\frac{C}{4+4C}\right)^{\nu} \frac{\nu!}{(r-i)!i!(\nu-r-j)!j!}$$

Where the prime on the summation means to count only even indices. Therefore

$$I_{A2} = \frac{\pi^2}{Cd^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{\nu=2m+2n}^{\infty}{}' \sum_{r=2m}^{\nu}{}' \left(\frac{C}{4+4C}\right)^{\nu} \frac{\nu!}{(r-i)!i!(\nu-r-j)!j!} \delta(k-2m)\delta(l-2n)$$

or in terms of λ

$$I_{A2} = \frac{\pi^2}{\lambda^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{\nu=2m+2n}^{\infty}{}' \sum_{r=2m}^{\nu}{}' \left(\frac{\lambda^2}{4(d^2+\lambda^2)}\right)^{\nu} \frac{\nu!}{(r-i)!i!(\nu-r-j)!j!} \delta(k-2m)\delta(l-2n).$$

VI. CONCLUSIONS

The basis of this thesis was to help the reader form a better understanding of the shallow water equations and also to see if an inverse Fourier transform for the semi-discretized system can give any information on how the system behaves as a function of time. The first four chapters consisted of background material that I compiled as a result of studying the shallow water equations, and also in an attempt to give the reader a logical order to better facilitate understanding of the later material. As can be seen, various forms of the shallow water equations were discussed and derived. Most of that work was taken directly from the references and were so noted. I feel this thesis will stand as a means for future students to more easily come to an understanding of shallow water equations and help them to pursue the next step in the process by finding a way of looking at other finite difference and finite element schemes.

I hope you enjoyed reading this thesis.

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